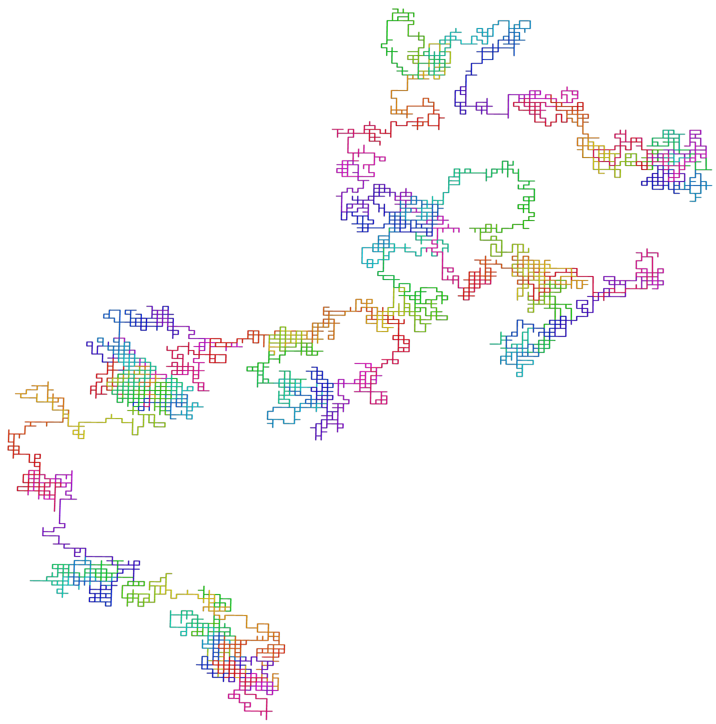
PROBABILITY THEORY

MATH 154

In class Final Exam

You have 45 minutes to complete the problems.
We start all together at 9 AM sharp. Do not open the booklet before that.
You can already fill in your name before that and contemplate the random walk in $\mathbb{Z}^2$ and worry whether it will return to the origin or not.



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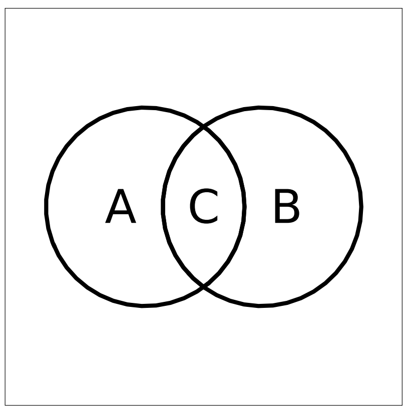
Name:

1) Check the boxes, where the operation $*$ has the property. A is an event in a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. The notation A^c means the complement.

$*$	$A * A^c = A$	$A * A^c = \emptyset$
Δ		
$\cap$		
$\cup$		
$\setminus$		

2) What is the relation between π-system and Λ-system and σ-algebra? Match A,B,C so that the Venn diagram works. Then answer whether the structure is a Boolean algebra.

	A, B, C	Is a Boolean algebra? yes or no
π system		
Λ system		
σ algebra		



3) Fill in the box.

A) $\sum_n \mathrm{P}[|X_n - X| \geq \epsilon] < \infty, \forall \epsilon > 0$
B) $\mathrm{P}[|X_n - X| \geq \epsilon] \to 0, \forall \epsilon > 0$
C) $\mathrm{P}[X_n \to X] = 1$
D) $||X_n - X||^2 \to 0.$
E) $F_{X_n}(t) \to F_X(t), \forall t \in \mathbb{R}.$

Convergence	Fill in A-E
$X_n \to X$ almost everywhere	
$X_n \to X$ in distribution	
$X_n \to X$ in $\mathcal{L}^2$	
$X_n \to X$ complete	
$X_n \to X$ in probability	

4) Which type of convergence appears in the following theorem? Reuse A-E from above.

Theorem	Fill in A-E from 3)
Von Neumann's ergodic theorem	
The golden theorem of Bernoulli	
The Central limit theorem	
The strong law of large numbers	
Birkhoff's ergodic theorem	

5) Fill into the box:

A) $h(c) \cdot \mathrm{P}[X \geq c] \leq \mathrm{E}[h(X)]$
B) $E[h(X)] \leq h(E[X])$.
C) $\mathrm{P}[X \geq c] \leq \inf_{t \geq 0} e^{-tc} M_X(t)$
D) $||X+Y||_p \leq ||X||_p + ||Y||_p$
E) $||XY||_1 \leq ||X||_2 \cdot ||Y||_2$
F) $D[p,q] = \sum_k p_k \log(\frac{p_k}{q_k}) \geq 0$
G) $\mathrm{P}[|X - \mathrm{E}[X]| \geq c] \leq \frac{\mathrm{Var}[X]}{c^2}$

Note that we sometimes called Chebychev-Markov simply Markov and we use Markov here to distinguish it formally from Chebychev.

Inequality	A-G or O if none applies
Cauchy-Schwarz	
Chernoff	
Minkowski	
Gibbs	
Markov	
Chebychev	
Jensen	

6) This entry is about stochastic processes. Sort the following system in decreasing generality. The order you give should be analog to ordering like animals $\supset$ insects $\supset$ ants. But use the labels a)-e). An example answer would be $a \supset b \supset c \supset d \supset e$

a) Bernoulli systems
b) Ergodic systems
c) Weakly mixing systems
d) Measure preserving systems
e) Mixing systems

7) We look at some proof techniques: which of the techniques were used to (i) prove the Central limit theorem or (ii) the statement that ergodicity is equivalent to constant random variables are the only T-invariant random variables, and (iii) the random walk theorem.

Technique	(i) central limit	(ii) Ergodicity	(iii) random walk
Jensen's inequality			
Proof by induction			
Proof by contradiction			
Generating functions			
Characteristic functions			

8) People: which of the following folks are associated (by name) to paradoxa in probability?

Person	Check if associated to a paradoxon
Dynkin	
Kronecker	
Venn	
Poisson	
Vitali	
Bernoulli	
Newcomb	
Von Neumann	
Bertrand	
Polya	
Markov	
Chernoff	
Poincare	
Riemann	
Kolmogorov	
Simpson	

9) Short answer questions

1) True or false? The Poincare recurrence theorem works even if T is not ergodic.

[ ]

2) The fact that a positive Markov matrix A has a unique eigenvalue 1 is called the [ ] theorem.

3) The random walk on the lattice has been proven by [ ].

4) A probability vector is a vector for which all entries add up to 1 and are [ ].

5) A Markov matrix has the property that all [ ] are probability vectors.

6) The central limit theorem for random variables X_i taking values 1 or 0 with probability p or $(1-p)$ is called the [ ] theorem.

7) If σ is the standard deviation of a sequence of IID random variables $X_1, \dots, X_n$, then $\sigma/\sqrt{n}$ is called the [ ].

8) A sequence of Binomial $B(n, 5/n)$ distributed random variables converges in distribution to the [ ] distribution.

9) What is the formula for differential entropy? [ ]

10) A measure preserving transformation is weakly mixing if and only if [ ] is ergodic.

10) Write down the names of 10 theorems we have proven in this course, but order according to your personal preferences! This is a little competition for the theorems. Oliver will announce which theorem is the winner.

Theorem	Give the precise name and if possible the person/persons
Number 1	
Number 2	
Number 3	
Number 4	
Number 5	
Number 6	
Number 7	
Number 8	
Number 9	
Number 10	

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