PROBABILITY THEORY

MATH 154

Final Exam Paper

Assignment: due Saturday May 9th.

Write an expository paper about a topic in probability theory of your choice. There are 40 suggestions here. You can go from one of them, modify or morph it into something else. If you plan to deviate much or suggest something else, please check with Oliver.

Rules

- The paper needs to be written by yourself. No human or AI assistance is allowed to write.
- Acknowledge any references like books, papers, web, AI chat, discussions, computer algebra.
- Aim for 4 or more pages with references. There can be illustrations (with credits if the illustration is not self-made).

Evaluation criteria

1) Mathematical correctness 2) Clarity, readability and elegance 3) Adaptation to the course, notation 4) References and sources 5) Originality or depth or surprise We will give 10 points for 1) and 2), 10 points for 3) and 4) and 5 points for 5) The quiz gives 5 points. There are maximally 30 points.

Topic suggestions

(1) **Probability in number theory.** Why does one expect to have the number of possibilities to write an even number 2n as a sum of two primes to behave like C log(n)? Why is the probability that two random numbers are coprime equal to $6/\pi^2$? Why does one expect the number of prime twins to grow like $n/\log(n)^2$?

(2) **Probability in physics.** What is statistical physics? How does probability theory enter quantum mechanics? Does god play dice? What is Brownian motion? What is temperature? What are critical phenomena?

(3) **Circular distributions** The Bayesian Central Limit Theorem (BCLT).

(4) **Probability theory in AI** How does probability theory enter modern large language models.

(5) **"Percolation"** Explore a result in the theory of percolation like the existence of a percolation threshold.

(6) **"Three series theorem"** of Kolmogorov about convergence of sums. Something related to the large law which we did not cover.
(7) **"Law of iterated log"** A refinement of the strong law of large numbers. We have seen that in the first homework.
(8) **"Risky central limit"** Central limit theorem with Cauchy fixed point theorem, for random variables with Césaro variance 1. It is a topic covered by Kolmogorov and Gnedenko.
(9) **"Random graphs"**. There are various models like Erdoes-Renyi graphs. Explore a mathematical result in this area.
(10) **"Oseledec's theorem"** for matrix valued random variables generalizes the Birkhoff ergodic theorem and is important in chaos thoory.
(11) **"Subadditive ergodic theorem"** Kingman's subadditive ergodic theorem generalizes the Birkhoff ergodic theorem.
(12) **"Riemann hypothesis."** Sufficient randomness of the Moebius μ function implies the Riemann hypothesis. We have seen that in the homework. Work this out more, or look at the history of the Mertens story.
(13) **"Point processes"**. Define Poisson point processes. We have not studied that but it is a topic with many applications especially with point processes in higher dimensions
(14) **"Vlasov systems"** The n-body problem is the ODE $\ddot{x} = \sum_{y \neq x} F(x, y)$. This generalizes from point measures to general measures.
(15) **"Normality of numbers."** An experimental mathematics topic: which numbers are normal. Is π is normal?
(16) **"Generic random variables"** A Baire generic random variable is neither absolutely continuous, nor discrete. (Similar to: a generic continuous function is nowhere differentiable). Baire category theory is a parallel story to probability but more topological.
(17) **"Noncommutative probability theory"** Probability of bounded random variables involves the commutative von Neumann algebra $\mathcal{L}$. Non-commutative measure theory is the theory of von Neumann algebras.
(18) **"Fuerstenbergs theorem"** Is related to the multiple recurrence theorem. This is a topic extending the recurrence theme.
(19) **"Unique ergodicity"** Uniquely ergodic systems have exactly one invariant probability measure. Examples are substitution systems or irrational rotations. There are also irrational rotations on higher dimensional tori.
(20) **"Random Jacobi operators"** are operator valued random variables. Their "law" is the **density of states**. Example: X_i process: Deine $L_{ij} = 1$ if $|i-j| = 1$ and $L_{ii} = X_i$ and $L_{ij} = 0$ for $|i-j| > 1$.
(21) **"Girko's law for Random matrices"** A random $n \times n$ matrix produces n random eigenvalues. What is the distribution in the complex plane?
(22) **"The Denjoy-Koksma theorem"** Describes the growth of Birkhoff sums S_n for Kronecker systems. You had looked at it already a bit in the problem set.
(23) **"Integral geometry"**. Applies probability theory in differential geometry. Example: Crofton formula for pi. One can measure lengths of curves or area of surfaces using probability theory.
(24) **"The Gauss-Kuzmin theorem".** The map $T(x) = 1/x \bmod 1$ preserves a measure such that T is ergodic.

(25) **"Pseudo random number generators"** How can we generate "random numbers" using deterministic processes. This topic is important in cryptology.

(26) **'Monte Carlo methods"** Probability theory computes things like integrals, area of the Mandelbrot set or quark masses.

(27) **"Brownian motion"** A prototype continuous time stochastic process. How do we define this?

(28) **"Ruin times"** First return to zero. Example: Ballot theorem for random walks.

(29) **"Higher dimensional Bertrand"** Hit n dimensional sphere randomly.

(30) **"Geometry of numbers"** Number of lattice points are there in a region?

(31) **"Directional data"** There is a central limit theorem for Lie group valued random variables.

(32) **"Subshifts of finite type"** produce interesting stochastic processes.

(33) **"Birkhoff sums"** S_n if $X_k = f(T^k x)$ for a measure preserving system. Example $T(x) = x + \alpha$ with golden mean α and $f(x) = \cot(\pi x)$.

(34) **Random Taylor series** If X_n are IID random variables, one can look at a random Taylor series $f(z) = \sum_{n=1}^{\infty} X_n z^n$.

(35) **"Random Dirichlet series"** If X_n are IID random variables, one can look at random Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} X_n/n^s$. This is Riemann Zeta if $var[X] = 0$.

(36) **"Random sphere packings"**. Compare with lattice packings.

(37) **"Diffusion limited aggregation"** is a model for pattern formation. Look at probabilistic connections.

(38) **"Ergodic theory of billiards"**. Especially Birkhoff billiards. Define the probability space and look what is known about ergodicity.

(39) **"Cryptological applications"**. Gleason's attack the Japanese Coral Cypher using a variant of the Chernov bound. An other example: the Pollard rho method to produce pseudo random numbers and exploit the Birthday paradox to get factors of integers.

(40) **"Martingale limit theorem"** A fresh approach to limit theorems is by using Martingales. "What is a martingale?"

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