Unit 24: Martingales

24.1. A sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of sub σ-algebras of $\mathcal{A}$ is called a **filtration**, if $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$. Given a filtration $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$, one gets a **filtered space** $(\Omega, \mathcal{A}, \{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathrm{P})$.

24.2. A **discrete time stochastic process** $X = \{X_n\}_{n\in\mathbb{N}}$ is called **adapted to a filtration** $\{\mathcal{A}_n\}$ if X_n is $\mathcal{A}_n$-measurable for all $n \in \mathbb{N}$.

24.3. A $\mathcal{L}^1$-process which is adapted to a filtration $\{\mathcal{A}_n\}$ is called a **martingale** if

$$\mathrm{E}[X_n|\mathcal{A}_{n-1}] = X_{n-1}$$

for all $n \geq 1$. It is called a **super-martingale** if $\mathrm{E}[X_n|\mathcal{A}_{n-1}] \leq X_{n-1}$ and a **sub-martingale** if

$$\mathrm{E}[X_n|\mathcal{A}_{n-1}] \geq X_{n-1} \ .$$

If we mean either sub-martingale or super-martingale (or martingale) we speak of a **semi-martingale**.

24.4. It immediately follows that for a martingale

$$\mathrm{E}[X_n|\mathcal{A}_m] = X_m$$

if $m < n$ and that $\mathrm{E}[X_n]$ is constant.

Allan Gut mentions in his book that a martingale is an allegory for ”life” itself: *the expected state of the future given the past history is equal the present state and on average, nothing happens.* The word “martingale” originally denoted a gambling system strategy in which losing bets are doubled. It is also the name of a part of a horse’s harness or a belt on the back of a man’s coat.

24.5. If a martingale X_n is given with respect to a filtered space $\mathcal{A}_n = \sigma(Y_0, \dots, Y_n)$, where Y_n is a given process, then X is called a **martingale with respect** Y.

24.6. If X is a super-martingale, then $-X$ is a sub-martingale and vice versa. A super-martingale, which is also a sub-martingale is a martingale. Since we can change X to $X - X_0$ without destroying any of the martingale properties, we could assume the process is **null at** 0 which means $X_0 = 0$.

24.7. Given a martingale. From the tower property of conditional expectation follows that for $m < n$

$$\mathrm{E}[X_n|\mathcal{A}_m] = \mathrm{E}[E[X_n|\mathcal{A}_{n-1}]|\mathcal{A}_m] = \mathrm{E}[X_{n-1}|\mathcal{A}_m] = \cdots = X_m \ .$$

24.8. Sum of independent random variables
Let $X_i \in \mathcal{L}^1$ be a sequence of independent random variables with mean $\mathrm{E}[X_i] = 0$. Define $S_0 = 0, S_n = \sum_{k=1}^n X_k$ and $\mathcal{A}_n = \sigma(X_1, \dots, X_n)$ with $\mathcal{A}_0 = \{\emptyset, \Omega\}$. Then S_n is a martingale since S_n is an $\{\mathcal{A}_n\}$-adapted $\mathcal{L}^1$-process and

$$\mathrm{E}[S_n|\mathcal{A}_{n-1}] = \mathrm{E}[S_{n-1}|\mathcal{A}_{n-1}] + \mathrm{E}[X_n|\mathcal{A}_{n-1}] = S_{n-1} + \mathrm{E}[X_n] = S_{n-1} \; .$$

We have used linearity and the independence property of the conditional expectation.

24.9. Example a) Conditional expectation
Given a random variable $X \in \mathcal{L}^1$ on a filtered space $(\Omega, \mathcal{A}, \{\mathcal{A}_n\}_{n \in \mathbb{N}}, \mathrm{P})$. Then $X_n = \mathrm{E}[X|\mathcal{A}_n]$ is a martingale.
Especially: given a sequence Y_n of random variables. Then $\mathcal{A}_n = \sigma(Y_0, \dots, Y_n)$ is a filtered space and $X_n = \mathrm{E}[X|Y_0, \dots, Y_n]$ is a martingale. Proof: by the tower property

$$\begin{aligned} \mathrm{E}[X_n|\mathcal{A}_{n-1}] &= \mathrm{E}[X_n|Y_0, \dots, Y_{n-1}] \\ &= \mathrm{E}[\mathrm{E}[X|Y_0, \dots, Y_n]|Y_0, \dots, Y_{n-1}] \\ &= \mathrm{E}[X|Y_0, \dots, Y_{n-1}] = X_{n-1} \; . \end{aligned}$$

verifying the martingale property $\mathrm{E}[X_n|\mathcal{A}_{n-1}] = X_{n-1}$.
We say X is a **martingale with respect to** Y. Note that because X_n is by definition $\sigma(Y_0, \dots, Y_n)$-measurable, there exist Borel measurable functions $h_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $X_n = h_n(Y_0, \dots, Y_{n-1})$.

24.10. Example b) Product of positive variables
Given a sequence Y_n of independent random variables $Y_n \geq 0$ satisfying with $\mathrm{E}[Y_n] = 1$. Define $X_0 = 1$ and $X_n = \prod_{i=0}^n Y_i$ and $\mathcal{A}_n = \sigma(Y_1, \dots, Y_n)$. Then X_n is a martingale. This is an exercise. Note that the martingale property does not follow directly by taking logarithms.

24.11. Example c) Product of matrix-valued random variables
Given a sequence of independent random variables Z_n with values in the group $GL(N, \mathbb{R})$ of invertible $N \times N$ matrices and let $\mathcal{A}_n = \sigma(Z_1, \dots, Z_n)$. Assume $\mathrm{E}[\log ||Z_n||] \leq 0$, if $||Z_n||$ denotes the norm of the matrix (the square root of the maximal eigenvalue of $Z_n \cdot Z_n^*$, where Z_n^* is the adjoint). Define the real-valued random variables $X_n = \log ||Z_1 \cdot Z_2 \cdots Z_n||$, where $\cdot$ denotes matrix multiplication. Because $X_n \leq \log ||Z_n|| + X_{n-1}$, we get

$$\begin{aligned} \mathrm{E}[X_n|\mathcal{A}_{n-1}] &\leq \mathrm{E}[\log ||Z_n|| \mid \mathcal{A}_{n-1}] + \mathrm{E}[X_{n-1}|\mathcal{A}_{n-1}] \\ &= \mathrm{E}[\log ||Z_n||] + X_{n-1} \leq X_{n-1} \end{aligned}$$

so that X_n is a super-martingale. In ergodic theory, such a matrix-valued process X_n is called **sub-additive**.

24.12. Example d) If Z_n is a sequence of matrix-valued random variables, we can also look at the sequence of random variables $Y_n = ||Z_1 \cdot Z_2 \cdots Z_n||$. If $\mathrm{E}[||Z_n||] = 1$, then Y_n is a super-martingale.