**PROBABILITY THEORY**

MATH 154

# Unit 21: Random walks

**21.1.** A **random walk** or **Markov chain** on an undirected graph $(V, E)$ is defined by a linear map which preserves probability vectors on $V$. The **adjacency matrix** $A$ of $(V, E)$ defines a **stochastic matrix** $M_{xy} = A_{xy}(x)/d(y)$, where $d(y)$ is the vertex degree of a vertex $y \in V$. The probability measures $M^n p(o)$ with initial probability measure $p(o)$ located on the initial point $o$. It is the distribution of the **standard random walk**. We return to the Markov picture next class.

**21.2.** If $(V, E)$ is the standard lattice $\mathbb{Z}^d$, then each vertex degree is $d(x) = 2d$. We can describe a path of a random walk by defining IID random vectors $X_i$ which take values in $I = \{e \in \mathbb{Z}^d \mid |e| = \sum_{i=1}^d |e_i| = 1 \}$ and which have the uniform distribution defined by $\mathrm{P}[X_n = e] = (2d)^{-1}$ for all $e \in I$. The random variable $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$ describes the position of the walker at time $n$. The stochastic process $S_n$ is called the **random walk** on the lattice $\mathbb{Z}^d$. The law of $S_n$ is a measure on $\mathbb{Z}^d$ which agrees with $M^n p(0)$.

**21.3.** Define the sets $A_n = \{S_n = 0 \}$ and the random variables $Y_n = 1_{A_n}$. If the walker has returned to position $0 \in \mathbb{Z}^d$ at time $n$, then $Y_n = 1$, otherwise $Y_n = 0$. The sum $B_n = \sum_{k=0}^n Y_k$ counts the number of visits of the origin 0 of the walker up to time $n$ and $B = \sum_{k=0}^\infty Y_k$ counts the total number of visits at the origin. The expectation

$$\mathrm{E}[B] = \sum_{n=0}^\infty \mathrm{P}[A_n]$$

tells us how many times the walker is expected to return to the origin. We write $\mathrm{E}[B] = \infty$ if the sum diverges. In this case, the walker returns back to the origin infinitely many times.

**Theorem 1** (Polya). *$\mathrm{E}[B] = \infty$ for $d = 1, 2$ and $\mathrm{E}[B] < \infty$ for $d > 2$.*

*Proof.* Fix $n \in \mathbb{N}$ and define $a^{(n)}(k) = \mathrm{P}[S_n = k]$ for $k \in \mathbb{Z}^d$. Because the walker can reach in time $n$ only a bounded region, the function $a^{(n)} : \mathbb{Z}^d \to \mathbb{R}$ is zero outside a bounded set. We can therefore define its Fourier transform

$$\phi_{S_n}(x) = \sum_{k \in \mathbb{Z}^d} a^{(n)}(k) e^{2\pi i k \cdot x}$$

which is a smooth function on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. It is the characteristic function of $S_n$ because

$$\mathrm{E}[e^{ixS_n}] = \sum_{k \in \mathbb{Z}^d} \mathrm{P}[S_n = k] e^{ik \cdot x} .$$

The characteristic function $\phi_X$ of $X_k$ is

$$\phi_X(x) = \frac{1}{2d} \sum_{|j|=1} e^{2\pi i x_j} = \frac{1}{d} \sum_{i=1}^{d} \cos(2\pi x_i) .$$

Because the $S_n$ is a sum of $n$ independent random variables $X_j$

$$\phi_{S_n} = \phi_{X_1}(x)\phi_{X_2}(x)\dots\phi_{X_n}(x) = \frac{1}{d^n}(\sum_{i=1}^{d} \cos(2\pi x_i))^n .$$

Note that $a_n(0) = \mathrm{P}[S_n = 0] = \int_{\mathbb{T}} \phi_{S_n}(x)\, dx$.
We now show that $\mathrm{E}[B] = \sum_{n \geq 0} \phi_{S_n}(0)$ is finite if and only if $d < 3$. The Fourier inversion formula using the normalized Volume measure $dx$ on $\mathbb{T}^3$ gives

$$\sum_n \mathrm{P}[S_n = 0] = \int_{\mathbb{T}^d} \sum_{n=0}^{\infty} \phi_X^n(x)\, dx = \int_{\mathbb{T}^d} \frac{1}{1 - \phi_X(x)}\, dx .$$

A Taylor expansion $\phi_X(x) = 1 - \sum_j \frac{x_j^2}{2}(2\pi)^2 + \dots$ shows

$$\frac{1}{2} \cdot \frac{(2\pi)^2}{2d}|x|^2 \leq 1 - \phi_X(x) \leq 2 \cdot \frac{(2\pi)^2}{2d}|x|^2 .$$

The claim of the theorem follows because the integral $\int_{\{|x|<\epsilon\}} \frac{1}{|x|^2}\, dx$ over the ball of radius $\epsilon$ in $\mathbb{R}^d$ is finite if and only if $d \geq 3$. ☐

**21.4.** We can now decide whether the random walker returns infinitely many times to 0 or not.

**Theorem 2.** *The walker returns to the origin infinitely often almost surely if* $d \leq 2$. *For* $d \geq 3$*, the walker almost surely returns only finitely many times and* $\mathrm{P}[\lim_{n\to\infty} |S_n| = \infty] = 1$.

*Proof.* If $d > 2$, then $A_\infty = \limsup_n A_n$ is the subset of $\Omega$, for which the particles returns to 0 infinitely many times. Since $\mathrm{E}[B] = \sum_{n=0}^{\infty} \mathrm{P}[A_n]$, the Borel-Cantelli lemma gives $\mathrm{P}[A_\infty] = 0$ for $d > 2$. The particle returns therefore back to 0 only finitely many times and in the same way it visits each lattice point only finitely many times. This means that the particle eventually leaves every bounded set and converges to infinity.
If $d \leq 2$, let $p = \mathrm{P}[\bigcup_n A_n]$ be the probability that the random walk returns to 0. Then $p^{m-1}$ is the probability that there are at least $m$ visits in 0 and the probability is $p^{m-1} - p^m = p^{m-1}(1-p)$ that there are exactly $m$ visits. We can write

$$\mathrm{E}[B] = \sum_{m \geq 1} m p^{m-1}(1-p) = \frac{1}{1-p} .$$

Because $\mathrm{E}[B] = \infty$, we know that $p = 1$. ☐