

PROBABILITY THEORY

MATH 154

Unit 3: Algebras

3.1. If Ω is a set, a set \mathcal{A} of subsets of Ω is called an **algebra** if it is closed under **intersection** \cap and **symmetric difference** Δ and if $\Omega \in \mathcal{A}$. The algebra of sets behaves like the algebra of integers. We just have to think about $\Delta = +$ as addition and $\cap = \cdot$ as multiplication. Indeed, we can check **commutativity**, **associativity** and **distributivity**. These identities are logical conclusions. We can also visualize them as **Venn diagrams**. Figure 2) shows associativity $(A \cdot B) \cdot C = A \cdot (B \cdot C)$. It just encodes the set of elements in Ω which are in all of the sets. The algebra encodes so basic logical thinking rules that usually are taken for granted. Boolean algebra includes also Boolean logic like the “tertium non datur” $A \cup A^c = \Omega$.

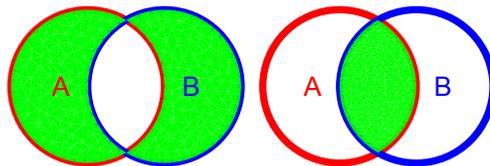


FIGURE 1. Venn diagram of the addition and multiplication in the algebra.

3.2. Any algebra of sets always contains the **empty set** $0 = \emptyset$, which is part of the algebra because it is $\Omega + \Omega$. The empty set is also called **zero** because $0 + A = A$ for every A . The set Ω plays the role of the 1 because $1 \cdot A = A$. The algebra is also known as a **Boolean algebra** because $A + A = 0$ and so $-A = A$. We can form other set operations like the union $A \cup B = AB + A + B = 1 + (1 + A)(1 + B)$ and the set difference $A \setminus B = B + AB$ and the **complement** $A^c = A + 1$. A Boolean algebra is a **commutative ring** with 1. Besides the laws $A + A = 0$ and $A^2 = A$, we have in particular $1 + 1 = 0$.

3.3. A set I is called **countable** if there is a bijection from I to the counting numbers $\mathbb{N} = \{1, 2, 3, \dots\}$. Every countable set of sets can therefore be written as a sequence of sets $\{A_1, A_2, \dots\}$. The **Hilbert hotel** pop-culture picture is the result that \mathbb{N} and $2\mathbb{N} = \{2, 4, 6, 8, 10, \dots\}$ have the same cardinality. We can also count the rationals \mathbb{Q} , as seen in class. We can not count the numbers in the interval $[0, 1]$ however as Cantor showed in his famous diagonal argument: just assume to have an enumeration and construct from this a new number that is different from each of the numbers: just change in number k the k 'th digit. In probability theory, countability plays a role. Remember the **Vitali set**.

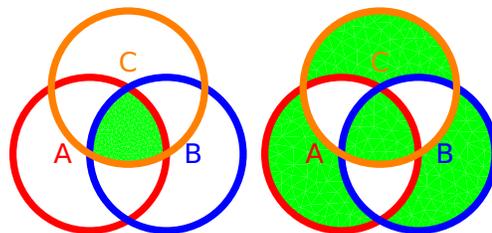


FIGURE 2. The Venn diagrams of multiplicative associativity $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ and additive associativity $A + (B + C) = (A + B) + C$.

3.4. An algebra is called σ -**algebra** if it is closed under the formation of countable unions. A pair (Ω, \mathcal{A}) , where \mathcal{A} is a σ -algebra on Ω is also called a **measurable space**. Formally, $A_n \in \mathcal{A} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$. This implies that countable intersections are in \mathcal{A} : $\bigcap_n A_n = 1 - (\bigcup_n (1 - A_n))$. An important property is that an arbitrary intersection of σ -algebras is a σ algebra.

3.5. Some examples:

- 1) For an arbitrary set Ω , $\mathcal{A} = \{\emptyset, \Omega\}$ is a σ -algebra. It is called the **trivial** σ -algebra.
- 2) If Ω is an arbitrary set, then $\mathcal{A} = 2^\Omega = \{A \subset \Omega\}$ is a σ -algebra. It is the largest σ -algebra one can define on Ω .
- 3) A finite set of pairwise disjoint sets A_1, A_2, \dots, A_n of Ω satisfying $\bigcup_j A_j = \Omega$ is called a **finite partition** of Ω . It generates the finite σ algebra $\mathcal{B} = \{A = \bigcup_{j \in J} A_j\}$, where J runs over all subsets of $\{1, \dots, n\}$. Every finite σ -algebra has this form and its elements $\{A_1, \dots, A_n\}$ are called the **atoms** of \mathcal{B} .
- 4) Given an arbitrary set \mathcal{C} of subsets in Ω , we can look at the intersection of all σ algebras which contain \mathcal{C} . It is called the σ -algebra **generated** by \mathcal{C} .
- 5) Given a topology \mathcal{O} on Ω , a set of subsets of Ω that contains \emptyset, Ω and is closed under finite intersections and arbitrary unions. The σ algebra generated by this topology is called the **Borel** σ algebra of the topology.

3.6. Write $A_n \nearrow A$ if $A_n \subset A_{n+1}$ and $\bigcup_n A_n = A$. We say A is a **limit**. (Ω, \mathcal{A}) is called a π -**system**, if \mathcal{A} contains \emptyset and \mathcal{A} is closed under intersections. (Ω, \mathcal{A}) is called a λ -**system** or Dynkin system if \mathcal{A} contains Ω , is closed under complements and closed under limits. Note that both π systems as well as λ systems do not need to be algebras.

Theorem 1. (Ω, \mathcal{A}) is a σ -algebra \Leftrightarrow if it is a π -system as well as a λ -system.

Proof. " \Rightarrow ": Just check that $A \setminus B = A \cup B + B$.

" \Leftarrow ": assume \mathcal{A} is both a π -system and a λ system. Given $A, B \in \mathcal{A}$. By definition we know that $A^c = \Omega \setminus A, B^c = \Omega \setminus B$ is in \mathcal{A} . The π -system property implies that $A \cup B = \Omega \setminus (A^c \cap B^c) \in \mathcal{A}$. Since complements can be formed we have $A + B = A \cup B \setminus A \cap B$ in \mathcal{A} . Given a sequence $A_n \in \mathcal{A}$. Define $B_n = \bigcup_{k=1}^n A_k \in \mathcal{A}$ and $A = \bigcup_n A_n$. Because $B_n \nearrow A$ we know that A is a limit and that $A \in \mathcal{A}$. This finishes the proof that \mathcal{A} is a σ -algebra. \square