PROBABILITY THEORY

MATH 154

Unit 2: Classical challenges

The Bertrand paradox

2.1. Bertrand asked in 1889, what the probability is that a random line on the unit disc intersects it with a length $\geq \sqrt{3}$, the length of the inscribed equilateral triangle. Here is the argument for $\mathrm{P} = 1/3$: fix a point A on the boundary of the disc we can look at all lines through that point. For a polar angle $0 < \theta < \pi/3$ and $2\pi/3 < \theta < \pi$ the chord is longer than $\sqrt{3}$ and for $\pi/3 < \theta < 2\pi 3$ it is larger. A second answer gives $\mathrm{P} = 1/2$: by looking at all points perpendicular to a fixed diameter the chord is longer than $\sqrt{3}$ if the point of intersection lies on the middle half of the diameter. A third answer gives $\mathrm{P} = 1/4$: if the midpoint of the chord is in the disc of radius $1/2$, the chord is longer than $\sqrt{3}$. The area of that disk is $1/4$ of the area of the disk.

The Monty-Hall paradox

2.2. Suppose you're on a game show and you are given a choice of three doors. Behind one door is a car and behind the others are goats. You pick a door-say No. 1 - and the host, who knows what's behind the doors, opens another door-say, No. 3-which has a goat. (In all games, he opens a door to reveal a goat). He then says to you, "Do you want to pick door No. 2?" (In all games he always offers an option to switch). Is it to your advantage to switch your choice?
No switching: you choose a door and win with probability 1/3. The opening of the host does not affect any more your choice. **Switching:** when choosing the door with the car, you loose since you switch. If you choose a door with a goat. The host opens the other door with the goat and you win. There are two such cases, where you win. The probability to win is 2/3. There are now entire books on the subject [**?**]. The problem is related to **Baysian thinking**.

The Banach Tarski paradox

2.3. We work in the probability space the unit cube Ω in $\mathbb{R}^3$, where the events are the set of all subsets of Ω and where the probability $\mathrm{P}[A]$ is the volume of A. We can take unions and intersections of events and keep having events. The axioms of probability theory assure that $\mathrm{P}[\Omega] = 1, \mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B]$ if A and B are disjoint. The volume of events is rotational and translational invariant as long as the turn or translation keeps us in Ω.

Now look at the following theorem: it is possible to write the ball $X = \{x^2 + y^2 + z^2 \leq 1/9\}$ as a disjoint union of 5 sets $X = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ and rotate and translate the sets in Ω to sets B_1, B_2, B_3, B_4, B_5 such that $B_1 \cup B_2 \cup B_3 = \{(x - 1/2)^2 + y^2 + z^2 \leq 1/9\} = X - (1/2, 0, 0)$ and $B_4 \cup B_5 = \{(x + 1/2)^2 + y^2 + z^2 \leq 1/9\} = X + (1/2, 0, 0)$. We have achieved, by cutting, translation and rotation that the ball has doubled in size. As $\mathrm{P}[S] + \mathrm{P}[S] = \mathrm{P}[S]$, this would imply $\mathrm{P}[S] = 0$. But since Ω can be covered by finitely many spheres of radius 1/3 and each having $\mathrm{P} = 0$, we conclude that $\mathrm{P}[\Omega] = 0$ which is a contradiction. We have not lied. The theorem follows from the Axiom of Choice and is true, the conclusion paradox was correctly derived. But there obviously will have been a problem.

The Petersburg paradox

2.4. Assume you pay an entrance fee c for a game and that you win 2^T, where T is the number of times, the casino flips a coin until "head" appears. For example, if the sequence of coin experiments would give "tail, tail, tail, head", you would win $2^3 - c = 8 - c$, the win minus the entrance fee. For which c is the game fair? We can compute the expectation as $\sum_{k=1}^{\infty} 2^k \mathrm{P}[T = k] = \sum_{k=1}^{\infty} 1 = \infty$. But nobody would agree to pay even an entrance fee $c = 20$. The event $T = 20$ is so improbable that it never occurs in the life-time of a person.

2.5. What would be a reasonable entrance fee in "real life"? Bernoulli proposed to replace the expectation $\mathrm{E}[G]$ of the profit $G = 2^T$ with the expectation of $(\mathrm{E}[u(G)])^2$, where $u(x)$ is a **utility function** like $(x) = \sqrt{x}$. It leads to a fair entrance estimate

$$(\mathrm{E}[\sqrt{G}])^2 = (\sum_{k=1}^{\infty} 2^{k/2} 2^{-k})^2 = \frac{1}{(\sqrt{2} - 1)^2} \sim 5.828... \,.$$

On the other hand, given any utility function $u(k)$, one can modify the casino rule. For example, we could just pay $(2^k)^2$ in the case $u(k) = \sqrt{k}$, or pay e^{2^k} for the utility function $u(k) = \log(k)$. Is there a good resolution to the difficulty?

The Martingale paradox

2.6. Here is a bullet proof **martingale strategy** in roulette: bet c dollars on red. If you win, stop, if you lose, bet $2c$ dollars on red. If you win, stop. If you lose, bet $4c$ dollars on red. Keep doubling the bet. Eventually after n steps, red will occur and you will win $2^n c - (c + 2c + \cdots + 2^{n-1} c) = c$ dollars. This example motivates the concept of martingales. Why can this foolproof strategy not used?

The Birthday paradox

2.7. The last example we mention in the notes illustrates that intuition can be misleading. There are 365 days in a year, so that that it appears that we appear to need a larger group of people to expect a Birthday collision. It turns out that already for a group with 23 people, the probability that two have the same birthday is larger than 1/2. Coincidences happen more frequently. In class, we will look at a "top 10" list of paradoxa including some not mentioned here.