

PROBABILITY THEORY

MATH 154

Homework 9

CENTRAL LIMIT

Problem 9.1: a) Find again the characteristic function ϕ_X of a standard Cauchy distributed random variable X . (We have done it before. Maybe try to do it without a computer algebra system using residue calculus.)
b) Deduce that if you take two independent standard Cauchy distributed random variables X, Y , then $(X + Y)/2$ is again standard Cauchy distributed.

Problem 9.2: a) Verify that the differential entropy of the Cauchy distribution with density $1/(\pi(1 + x^2))$ is $\log(4\pi)$. Mathematica gives wrongly $\log((1 + \sqrt{\pi})^2 \pi)$!
b) As a flashback, recall how the expectation $E[X]$ of a Cauchy distribution X is defined in a renormalized way by subtracting two infinite quantities.
c) Verify that the renormalized variance $\lim_{n \rightarrow \infty} \frac{1}{n} \int_{-n}^n x^2 f(x) dx$ exists for the Cauchy distribution. What is its value?

Problem 9.3: a) Compute the entropy of the standard distribution $N(0, 1)$. We have sketched it in class.
b) What is bigger, the entropy of the Cauchy distribution or the entropy of the standard normal distribution? c) Compute the entropy of the exponential distribution on $[0, \infty)$.

Problem 9.4: We work here with measures on $(\mathbb{R}, \mathcal{B})$.

a) Assume $d\mu(x) = f(x) dx$ and $d\nu(x) = g(x) dx$ are absolutely continuous probability measures. The convolution $f * g(x) = \int_{\mathbb{R}} f(y)g(x - y) dy$ defines a new measure $d\mu * d\nu = f * g dx$. Verify

$$\int f * gh(z) dz = \int \int h(x + y)f(y) dyg(z) dz .$$

b) Conclude that

$$d\mu * d\nu(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x + y) d\mu(x) d\nu(y)$$

c) Verify that the transformation

$$T_\lambda(\mu)(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A\left(\frac{x + y}{\sqrt{2}}\right) d\mu(x) d\mu(y)$$

on the space of all Borel probability measures on $(\mathbb{R}, \mathcal{B})$ satisfying $\int x d\mu(x) = 0$ has a unique fixed point.

Problem 9.5: We have seen that the central limit theorem implies the de Moivre central limit theorem so that in principle we do not need to prove it again. Write down a proof of the de Moivre central limit theorem. You have the following options: a) using the Stirling approximation formula $n! \sim \sqrt{2\pi n}(n/e)^n$ for the factorial.

b) using characteristic functions, essentially repeating the general proof.