PROBABILITY THEORY

MATH 154

Homework 8

Transformation

Problem 8.1: a) Check that the automorphisms of a probability space form a group. There is a subset of ergodic automorphisms. Investigate whether (i) ergodic, (ii) weakly mixing, (iii) mixing automorphisms form a subgroup.
b) For every $T \in \mathrm{Aut}(\Omega, \mathcal{A}, \mathrm{P})$ we have a unitary transformation $U : \mathcal{L}^2 \to \mathcal{L}^2$ given by $Uf = f(T)$. Check the orthogonality condition $\langle Uf, Ug \rangle = \langle f, g \rangle$.
c) Classical mechanics is the theory of automorphisms of probability spaces, where the unitary evolution is given by a dynamics $Uf = f(T)$. Quantum mechanics allows for a larger automorphism group consisting of all unitary operator $Uf = e^{itA} f$ with a self-adjoint operator A on the Hilbert space $\mathcal{L}^2(\Omega)$. Assume our probability space is finite. What is its classical automorphism group? What is its quantum automorphism group?

Problem 8.2: Show that if a measure-preserving transformation T has the property that for any $A, B \in \mathcal{A}$ there is m such that $\mathrm{P}[A \cap T^{-n}(B)] = \mathrm{P}[A]\mathrm{P}[B]$ for all $n \geq m$, then $\mathcal{A}$ is a trivial algebra.

Ergodicity

Problem 8.3: Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space, and let $T : \Omega \to \Omega$ be a measure-preserving transformation. Verify that the following conditions are equivalent:
(i) T is ergodic
(ii) If $A \in \mathcal{A}$ and $\mathrm{P}[T^{-1}(A) \Delta A] = 0$, then $\mathrm{P}[A] = 0$ or $\mathrm{P}[A] = 1$.
(iii) If $A \in \mathcal{A}$ satisfies $\mathrm{P}[A] > 0$ then $\mathrm{P}[\bigcup_n T^{-n}(A)] = 1$.
(iv) If $A, B \in \mathcal{A}$ satisfy $\mathrm{P}[A] > 0, \mathrm{P}[B] > 0$ then there is n such that $\mathrm{P}[T^{-n}(A) \cap B] > 0$.
Instead of checking all 12 possible ordered pairs, use the Merry-Go-Round proof technique: $(i) \to (ii) \to (iii) \to (iv) \to (i)$.

Proof. $\boxed{(i) \to (ii)}$ $P[T^{-1}(A)\Delta A] = 0$ means that $T(A) = A$ up to a measure zero. By definition A has measure 0 or 1. $\boxed{(ii) \to (iii)}$ The set $B = \bigcup_n T^{-n}(A)$ is invariant and so has measure 0 or 1. Since it contains A which has positive measure, it has measure 1.
$\boxed{(iii) \to (iv)}$ If there existed a set B which never can be reached, then B would be disjoint of $T^{-n}(A)$. But $P[\bigcup_n T^{-n}(A)] = 1$.
$\boxed{(iv) \to (i)}$ Assume $T^{-1}(A) = A$ and A has measure different from one. Then take $B = A^c$. □

Weak mixing

Problem 8.4: a) In the proof showing that T is mixing implies T^2 is mixing, we use the following Lemma from calculus or real analysis: the following two things are equivalent:
(i) $c_n \geq 0$ is a bounded sequence with $\frac{1}{n}\sum_{k=1}^n |c_k| \to 0$.
(ii) There exists a set J of density 1 in $\mathbb{N}$ on which $\lim_{j \in J} |c_k| \to 0$.
b) Use a) to verify that if $c_n \geq 0$ is a bounded sequence $\frac{1}{n}\sum_{k=1}^n |c_k| \to 0$ is equivalent to $\frac{1}{n}\sum_{k-1}^n |c_k|^2 \to 0$.
c) Conclude that weakly mixing can be rephrased as the property $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |P[A \cap T^{-k}(B)] - P[A]P[B]|^2 = 0$ for all $A, B \in \mathcal{A}$.

Mixing

Problem 8.5: a) Prove the following result of Rényi: A dynamical system T is mixing if and only if $\mu(A \cap T^{-n}A) \to \mu(A)^2$ for $n \to \infty$.
b) State and give a proof of the Riemann-Lebesgue lemma. Why does this lemma imply that T has only absolutely continuous spectrum, then T is mixing? (Use a).