PROBABILITY THEORY

MATH 154

Homework 7

Strong Law and Birkhoff

Problem 7.1: Let $(\Omega = [0,1]^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}}, \mathrm{P}^{\mathbb{N}})$ denote the standard product Lebesgue probability space. Consider for $n \geq 1$ the sequence $a_n = 1/(n\log(n))$. Define $X_n(x) = n1_{[0,a_n/2](x_n)} - n1_{[1-a_n/2,1](x_n)}$. In other words, we have a sequence of random variables that take values $n, -n, 0$.
a) Check that X_n is a sequence of independent random variables of zero mean and variance $n/(\log(n))$.
b) Check that the proof of the weak law of large numbers still works so that $\mathrm{P}[S_n/n \geq \epsilon] \to 0$.
c) Verify that $\sum_n \mathrm{P}[\{X_n = n\}]$ diverges and conclude that with probability 1, we have $|S_n/n| \geq 1/2$ infinitely many often.
d) Conclude that X_n does not satisfy the strong law of large numbers.

Problem 7.2: Use the notes to write down the proof of the maximal ergodic theorem of Hopf. Make sure you understand every step.

Problem 7.3: Use the notes to write down the proof of the Birkhoff ergodic theorem. Make sure you understand every step.

Problem 7.4: Write down a paragraph about the history of Birkhoff's ergodic theorem. Especially make a connection with Harvard.

Problem 7.5: Given a real number α let $T : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{T}$ be defined as $T(x) = x + \alpha$. A continuous function $f : \mathbb{T} \to \mathbb{R}$ defines so a sequence of random variables $X_n(x) = f(T^n(x)) = f(x + n\alpha)$.
a) If there exists a continuous function g such that $f(x) = g(x+\alpha) - g(x)$, we call f a coboundary). What can you say about the growth rate of S_n if f is a coboundary?
b) The sum S_n is also known as a Weyl sum. Assume f is continuous with $\int_0^1 f(x)\ dx = 0$ and that α is irrational. What does the Birkhoff ergodic theorem say about S_n/n?
c) Assume α is irrational. Are the random variables X_n independent? Are the random variables decorrelated? Can you use the strong law of large numbers to estimate S_n? Can you use the weak law of large numbers to estimate S_n?
d) Look up what happens if α has the Diophantine property $|\alpha - p/q| \leq 1/q^2$ for all rational numbers p/q. (An example is if α is the golden mean.) There is a result that assures that S_n stays bounded in this Diophantine case if f is continuous. Find that result and state it.