PROBABILITY THEORY

MATH 154

Homework 4

Independence

Problem 4.1: Ana and Bob own a business OAP (a pun to $(\Omega, \mathcal{A}, \mathrm{P})$) which builds custom designed dice. The customer wants a dice with a given probability distribution. OAP delivers 3D printed dice.
Ana spends 40 percent of her day in meetings, while Bob spends 25 percent of his day in meetings. They schedule their meetings independently.
a) What is the probability that both meet at the same time?
b) What is the probability that Ana has a meeting during a time that Bob has a meeting?
c) What is the probability that Bob has a meeting during a time when Ana has a meeting?
d) Is the event that both have a meeting at the same time independent of the event that both have no meeting at the same time?

FIGURE 1. Palindromes Ana and Bob meet to discuss the design of new dice. (AI generated picture)

Problem 4.2: True or False? (Please give justifications).
1) If A, B are independent, then A, B^c are independent.
2) If $\mathrm{P}[B] > 0$, and A, B are independent, then $\mathrm{P}[A|B] = \mathrm{P}[A]$.
3) If A, B are independent and B, C are independent then A, C are.
4) If A, B, C are independent, then $A + B$ is independent of C.
5) If A, B, C are independent, then $A \cap B$ is independent of C.
6) If A, B, C are independent then $A \cup B$ is independent of C.
7) Two disjoint sets A, B are independent if and only if $P[A] = 0$ or $P[B] = 0$.
8) $\emptyset$ is independent of any other set.
9) Ω is independent of any other set.
10) If A is independent to itself, then $\mathrm{P}[A] = 0$ or $\mathrm{P}[A] = 1$.

Problem 4.3: If $(\Omega, \mathcal{A}, \mathrm{P})$ has a $\mathcal{P}$ trivial σ-algebra, you might think that $\mathcal{A}$ is the trivial σ-algebra. This is not the case as you verify here with an example:
Verify that the σ algebra of cocountable or countable sets in $\Omega = [0, 1]$ is $\mathcal{P}$-trivial, if $\mathcal{P} = \lambda$ is the probability Lebesgue measure on $[0, 1]$

Problem 4.4: In all of this problem, all random variables are bounded $\mathcal{L}^\infty$.
a) Verify that if X, Y are independent and n, m are positive integers, then X^n, Y^m are independent.
b) Verify that $X \cdot Y = \langle X, Y \rangle = \mathrm{E}[XY]$ defines an inner product on $\mathcal{L}^2$. Define $|X| = \sqrt{\langle X, X \rangle}$. Check **Cauchy-Schwarz** $|\langle X, Y \rangle| \leq |X||Y|$.
c) We have seen that if X, Y are independent $\mathcal{L}^2$ random variables, then $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$. Can you reverse this? Does the condition $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$ imply that X, Y are independent?
d) What about asking that $\mathrm{E}[X^n Y^m] = \mathrm{E}[X^n]\mathrm{E}[Y^m]$ for all $n, m > 0$? Does this imply that X, Y are independent?

Problem 4.5: a) Verify that the moment generating function of the Cauchy distribution does not exist.
b) Compute the characteristic function $\phi_X(t)$ of a Cauchy distributed random variable.
c) Compute the characteristic function of the Gaussian distribution with probability density function $f(x) = e^{-x^2}/\sqrt{\pi}$.
d) Find a probability space and a random variable X such that $\phi_X(t) = \cos(t)$.