PROBABILITY THEORY

MATH 154

Final Exam Paper

Assignment: due Saturday May 10th.

Write an expository paper about a topic in probability theory of your choice.

Rules

- The paper is written by yourself. No human or AI assistance is allowed to write.
- Acknowledge any references: books, papers, web, AI chat, discussions, CAS.
- Aim for 4 or more pages with references. There can be illustrations. Credit illustrations from others.

Evaluation criteria

1) Mathematical correctness 2) Clarity, readability and elegance 3) References and sources 4) Adaptation to the course, notation 5) Originality or depth or surprise We will give 10 points for 1) and 2), 10 points for 3) and 4) and 5 points for 5) The quiz gives 5 points. There are maximally 30 points.

Topic suggestions

If you prefer an other topic you need to check with Oliver.

(1) **"Random graphs"**. Various models exist. i.e. Erdoes-Renyi
(2) **"Percolation"** Explore a result in the theory of percolation.
(3) **"Three series theorem"** of Kolmogorov about convergence of sums.
(4) **"Law of iterated log"** A refinement of the strong law of large numbers.
(5) **"Risky central limit"** Central limit theorem with Cauchy fixed point theorem. for random variables with Césaro variance 1.
(6) **"Oseledec's theorem"** For matrix valued random variables. Generalizes the Birkhoff ergodic theorem.
(7) **"Subadditive ergodic theorem"** Kingman's subadditive ergodic theorem generalizes the Birkhoff ergodic theorem.
(8) **"Riemann hypothesis."** Sufficient randomness of the Moebius μ function implies the Riemann hypothesis.
(9) **"Point processes"**. Define Poisson point processes. (We have seen the Chewing gum example)
(10) **"Vlasov systems"** The n-body problem is the ODE $\ddot{x} = \sum_{y \neq x} F(x, y)$. This generalizes from point measures to general measures.

(11) **"Is Pi normal?"** An experimental mathematics topic: π is normal?
(12) **"Generic random variables"** A Baire generic random variable is neither absolutely continuous, nor discrete.
(13) **"Noncommutative probability theory"** Probability of bounded random variables involves the commutative von Neumann algebra $\mathcal{L}$. Non-commutative measure theory is the theory of von Neumann algebras.
(14) **"Fuerstenbergs theorem"** Is related to the multiple recurrence theorem.
(15) **"Unique ergodicity"** Uniquely ergodic systems have exactly one invariant probability measure. Examples are substitution systems.
(16) **"Random Jacobi operators"** are operator valued random variables. Their "law" is the **density of states**. Example: X_i process: Deine $L_{ij} = 1$ if $|i-j| = 1$ and $L_{ii} = X_i$ and $L_{ij} = 0$ for $|i-j| > 1$.
(17) **"Girko's law for Random matrices"** A random $n \times n$ matrix produces n random eigenvalues. What is the distribution in the complex plane?
(18) **"The Denjoy-Koksma theorem"** Describes the growth of Birkhoff sums S_n for Kronecker systems
(19) **"Integral geometry"**. Applies probability theory in differential geometry. Example: Crofton formula.
(20) **"The Gauss-Kuzmin theorem".** The map $T(x) = 1/x \bmod 1$ preserves a measure such that T is ergodic.
(21) **"Pseudo random number generators"** How to get "random numbers".
(22) **'Monte Carlo methods"** Probability theory computes things like integrals, area of the Mandelbrot set or quark masses.
(23) **"Brownian motion"** A prototype continuous time stochastic process.
(24) **"Ruin times"** First return to zero. Example: Ballot theorem.
(25) **"Higher dimensional Bertrand"** Hit n dimensional sphere randomly.
(26) **"Geometry of numbers"** Number of lattice points are there in a region?
(27) **"Directional data"** There is a central limit theorem for Lie group valued random variables.
(28) **"Subshifts of finite type"** produce interesting stochastic processes.
(29) **"Birkhoff sums"** S_n if $X_k = f(T^k x)$ for a measure preserving system. Example $T(x) = x + \alpha$ with golden mean α and $f(x) = \cot(\pi x)$.
(30) **Random Taylor series** If X_n are IID random variables, one can look at a random Taylor series $f(z) = \sum_{n=1}^{\infty} X_n z^n$.
(31) **"Random Dirichlet series"** If X_n are IID random variables, one can look at random Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} X_n/n^s$. This is Riemann Zeta if $var[X] = 0$.
(32) **"Random sphere packings"**. Compare with lattice packings.
(33) **"Ergodic theory of billiards"**. Especially Birkhoff billiards.
(34) **"Cryptological applications"**. i.e. Gleason attacked the Japanese Coral Cypher using a variant of the Chernov bound.
(35) **"Martingale limit theorem"** A fresh approach to limit theorems.