

DIFFERENTIAL GEOMETRY

MATH 136

Midterm part II. Due Friday 10/16/2026, midnight

Handwritten. Write clearly. No internet, no correspondence, no computer algebra system or AI. This is closed book, Only one single double page of handwritten notes is allowed. Put your name on each page of your paper and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."**

Problem A (a) 5 points): Which of the following 5 graphs $G = (V, E)$ are discrete 2-manifolds? In each case, decide and also reason.

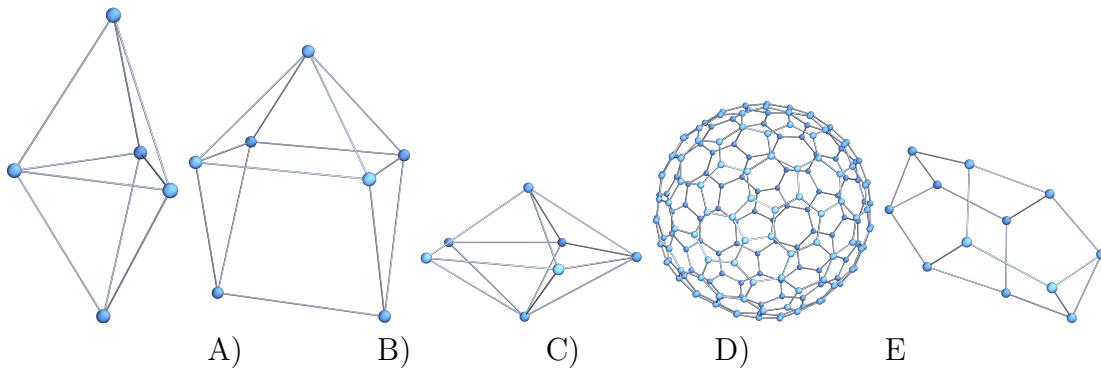


FIGURE 1. Manifold or not?

Problem A (b) 5 points): Fill in the values of the Euler characteristic in the following cases.

The sphere	Torus	Klein Bottle	Projective plane	Connected sum of two tori

Problem B (10 points): Lets look at the **nodal cubic curve** given as the contour curve

$$f(x, y) = y^2 - x^3 - x^2 = 0 .$$

An other elliptic curve is given as a parametrization

$$r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t^2 - 1 \\ t^3 - t \end{bmatrix} .$$

- (2 points) Check that the two cubics are the same.
- (2 points) Compute the Jacobian df to see where the curve is regular.
- (2 points) Compute the Jacobian dr to see where the curve is regular.
- (2 points) Compute $g = dr^T dr$ for the curve and write down the arc length integral from $t \in [-1, 1]$. Do not try evaluating it! It would produce elliptic integrals.
- (2 points) Compute the curvature value at the point $(-1, 0)$.

Problem C (10 points): The **Reuleaux triangle** is a curve of constant thickness. These shapes appear as guitar picks, dog tags, window shapes, or the shape of the Kresge auditorium at MIT. We have seen the Hopf Umlaufsatz in the continuum as well as in the discrete in a PSet.

- (2 points) What does the Hopf Umlaufsatz tell if you transverse a curve counter clockwise?
- (2 points) Formulate also the polygonal Umlaufsatz from the Pset in terms of angles.
- (2 points) Formulate now a new hybrid version, which combines both theorems.
- (2 points) Check your new theorem in the case of the Reuleux triangle seen left in the picture
- (2 points) Check your new theorem for half circles are placed over each side of the triangle seen right in the figure.

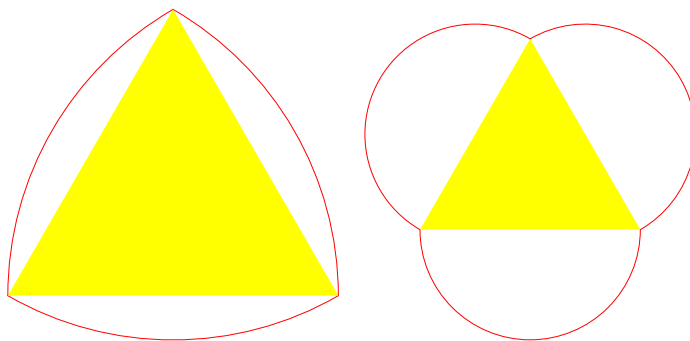


FIGURE 2. Reuleux triangle in part d) and a non-convex piecewise circular bubble in part e). The later region produces a chaotic billiard. Both regions are built using an equilateral triangle.

Problem D (10 points): Define the surface by the parametrization

$$r(u, v) = \begin{bmatrix} v \cos(u) \\ v \sin(u) \\ 2 \end{bmatrix}.$$

The following computations are by purpose not difficult and may be conceived easy. Even if things should be simple to you, give details of all definitions and computations!

- a) (2 points) Write down the first fundamental form $g = I$. In general and in this case.
- b) (2 points) Write down the second fundamental form $h = II$. In general and in this case.
- c) (2 points) Write down the third fundamental form $e = III$. In general and in this case.
- d) (2 points) How is the shape operator A defined from I, II ? In general and in this case.
- e) (2 points) How are K and H obtained, if you know I, II, III ? In general and in this case. Check the identity $III - 2HII + KI = 0$ in this case.

Problem E (10 points): We prove here a Gauss-Bonnet theorem for 3-dimensional convex 3-dimensional hyper-surfaces M in $\mathbb{R}^4$. The theorem tells that $\boxed{\iiint_M K \, dV = 2\pi^2}$. Your task is to complete the prove that $\iiint_M K \, dV$ is always the volume of the 3-dimensional sphere. (Not even your 200 level concentrator buddies might know about it.) You are told that $2\pi^2$ is the volume of the unit sphere $S^3 \sim SU(2)$. No need to prove this.

a) (2 points) The definitions are the same than for 2-manifolds. Please restate these definitions and also remind what dr and dn mean.

$$I = \boxed{} \cdot dr \quad II = \boxed{} \cdot dr$$

$$III = \boxed{} \cdot dn$$

b) (2 points) $A = \boxed{}$ $K = \boxed{}$

c) (2 points) Start with the definition $dn = -drA$, then proceed as we did in class to get $III = IIA$ and $III = IA^2$. Explain all three steps.

d) (2 points) Conclude, why this implies $\sqrt{\det(III)} = K\sqrt{\det I}$.

e) (2 points) As a sweet desert to this lovely math dinner, fill in the rest:

$$\begin{aligned} 2\pi^2 &= \iiint_R \sqrt{\det(\boxed{})} \, dudvdw \\ &= \iiint_R K \sqrt{\det(\boxed{})} \, dudvdw \\ &= \iiint_M K \, dV . \end{aligned}$$

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