DIFFERENTIAL GEOMETRY

MATH 136

Unit 21: Relativity

21.1. The **principle of general covariance** states that the pseudo Riemannian manifold (M, g) defines the laws of gravity. No preferred basis nor background "aether" concepts are allowed. and physical laws are invariant under smooth coordinate changes. Mathematically, this means that objects of interests are tensorial. Einsteins theory of gravity links **space-time** with **matter**. Matter determines space time (M, g) in that g by minimizing the **Hilbert action**. Space-time determines the paths of particles by minimizing **kinetic action**. [1]

The Einstein Equations	The Geodesic Equations
$\boxed{R - \frac{1}{2}Sg + \Lambda g = \kappa T}$	$\boxed{\ddot{x} = -\sum_{i,j} \Gamma_{ij}^k \dot{x}^i \dot{x}^j}$
Matter tells Space-time how to curve	Space-time tells Matter how to move

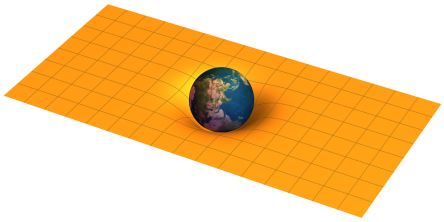


FIGURE 1. Mass deforms space time.

21.2. The theory has been experimentally confirmed in various instances like 1) The **perihelion advance of planets like Mercury**, 2) the **gravitational lensing of light** around stars or galaxies, 3) the **time delay in radar probing of planets**, 4) the **spectral shift of light emanating from massive objects**, 5) the **precession of a gyroscope**,freely orbiting the earth, 6) the **orbital decay of binary pulsars**, 7) the **detection of gravitational waves** from Black-hole mergers, 8) the **pictures of black holes** like Sagittarius A* and Messier 87* by the **event horizon telescope**.

[1] Even when ignoring any quantum mechanics, this is unsatisfactory. In a 2-body problem of two massive particles like a black-hole binary, the masses contribute to the stress-energy tensor T and make it singular. About the math: Yvonne Choquet-Bruhat (1923-2025) dealt with the Cauchy problem in 1969. Demetrios Christodoulou and Sergiu Klainermann proved in 1994 nonlinear gravitational stability. Numerical schemes for black hole binaries in vacuum using **post-Newtonian expansions** exist since 2005. It is ugly, both from an engineering as well as mathematical point of view.

21.3. In **special relativity**, the metric g is no more positive definite. The **Galilei group** generated by rotations and translations is replaced by the **Poincaré group** generated by **Lorentz transformations** and translations. Any rotation matrix in a (e_0, e_i)-plane with $i = 1, 2, 3$ is replaced by a **hyperbolic rotation**, where cos is replaced by cosh and sin by sinh leading to **Lorentz boosts**. For $M = \mathbb{R}^4$

$$g = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

with speed of light c is called the **flat Lorentz metric**. [2] The **null vectors** are the vectors in the **light cone** $\{||u|| = 0\}$. Vectors v satisfying $\langle v, v\rangle = \sum_{i,j} g_{ij} v^i v^j < 0$ are **time like**. Vectors $\langle v, v\rangle > 0$ are **space like**. Particles with space like velocity vectors have velocity smaller than the speed of light. Particles with time like velocity vectors are called **tachions**. They have velocity larger than the speed of light.

21.4. Important examples in GR is the **Schwarzschild metric** and the **Kerr metric**. All known direct experimental confirmations of GR are based on these models. On the manifold $M = \mathbb{R}^4 \setminus \{r \leq 2m\}$, we can use spherical coordinates $t = x^0, r = x^1, \theta = x^2, \phi = x^3$. For $r > 2m$, the **Schwarzschild** metric describing a **single massive object** of mass Mass. Write $m = G\text{Mass}/c^2$, where G is the gravitational constant.

$$g_S = \begin{bmatrix} -c^2(1-\frac{2m}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2m}{r})^{-1}) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}, g_K = \begin{bmatrix} -c^2(1-\frac{r_s r}{\sigma}) & 0 & 0 & \frac{r_s r a c \sin^2(\theta)}{\sigma} \\ 0 & \frac{\sigma}{\delta} & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ \frac{r_s r a c \sin^2(\theta)}{\sigma} & 0 & 0 & (r^2 + a^2 + \frac{r_s r a^2 \sin^2(\theta)}{\sigma})\sin^2(\theta) \end{bmatrix} .$$

is the **Kerr metric** describing a rotating axially symmetric black hole with Kerr parameter $a = J/(\text{Mass} \cdot c)$, Schwarzschild radius $r_s = 2m$, and $\sigma = r^2 + a^2\cos(\theta)^2$, $\delta = r^2 - r_s r + a^2$. For $J = 0$, it reduces to the diagonal Schwarzschild metric.
[3] The **Schwarzschild radius** $r_s = 2m$ is about 3km for the sun, less than $1cm$ for the earth). You check in a Pset that (M, g_S) and some other models satisfy the Einstein equations. We will verify g_K in class.

21.5. The following establishes why the theory "makes sense". GR is a refinement of **Newton's theory of gravitation**.

Theorem 1. *Relativity for slow particles in a weak field becomes Newtonian mechanics.*

Proof. Write $g = \overline{g} + h$, where $\overline{g} = Diag(-c^2, 1, 1, 1)$ is the flat metric and h is small. Velocities $\dot{x}^i$ for $i = 1, 2, 3$ can be neglected in comparison to $\dot{x}^0$. The geodesic equations $\ddot{x}^k = -\sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j$ simplify. Linearization gives $-\Gamma_{00k} = \partial_{u^k} \overline{g}_{00} - \partial_{u^0} h_{0k}$. If the gravitational field is time $u^0 = t$ independent, the second term is zero. We have now as contravariant vectors $\ddot{x} = -\frac{1}{2}\nabla h_{00}$. This is the **Newton equation** for $h_{00} = 2V/c^2$ with $V = m/r$. □

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[2] Like Misner,Thorne, and Wheeler, we assume g to be positive definite on **space-like** hypersurfaces. It has signature $-+++$.

[3] By a **theorem of Birkhoff**, Schwarzschild is the unique spherically symmetric solution of Einstein's equations. The no-hair theorem states that a stable stationary black hole can be described by mass, charge and angular momentum alone.