DIFFERENTIAL GEOMETRY

MATH 136

Unit 19: Discrete Manifolds

19.1. A **discrete m-manifold** is a finite graph $G = (V, E)$ for which every unit sphere $S(v)$ is a discrete $(m-1)$-sphere. A **discrete m-sphere** is a discrete m-manifold which has the property that removing a point renders it contractible. Inductively, a graph is called **contractible**, if both $S(v)$ and $S \backslash v$ are contractible for some $v \in V$. The 1-point space 1 is contractible. The empty graph is the (-1)-sphere. Let F_k denote the set of K_{k+1} subgraphs (k-simplices) and $f_k = |F_k|$. We have $F_0 = V, F_1 = E$. The **Euler characteristic** of M is defined as $\chi(M) = \sum_{k=0}^{m} (-1)^k f_k = f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^m f_m$. This definition of Ludwig Schläfli generalizes $\chi(M) = f_0 - f_1 + f_2 = V - E + F$ for 2-manifolds.

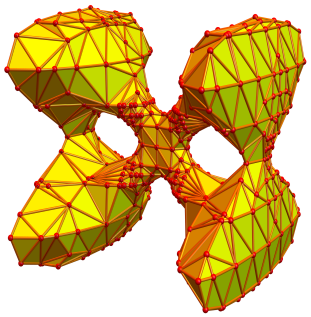


FIGURE 1. This 2-manifold M of genus $g = 2$ has $\chi(M) = 2 - 2g = -2$.

19.2. A graph without edges is a 0-manifold. A 0-manifold with 2 points is a 0-sphere. The reason is that removing a vertex produces the contractible K_1. Every connected 1-manifold is a 1-sphere, a circular graph C_n with $n \geq 4$. Every finite 2-manifold is either a 2-sphere S^2 or a connected sum of tori or projective planes: $M = S^2, M = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ or $M = \mathbb{P}^2 \# \cdots \# \mathbb{P}^2$. A 2-sphere can be characterized as 2-manifold of Euler characteristic 2. The 16 cell and the 600 cells are examples of 3-spheres. The join of two 1-spheres is a 3-sphere. The join of a k-sphere with a m-sphere is a $(k+m+1)$-sphere. The join of G with the 0-sphere is called **suspension**.

19.3. Euler's formula $\chi(M) = V - E + F = 2$ for 2-spheres generalizes to higher dimension. The 0-sphere has $\chi(M) = V = 2$, every 1-sphere has $\chi(M) = V - E = 0$. Every 2-sphere has $\chi(M) = V - E + F = 2$. This pattern continues:

Theorem 1 (Euler's Gem)**.** *If M is a m-sphere, then $\chi(M) = 1 + (-1)^m$.*

Proof. Use induction with respect to dimension m. For $m = 0$, we have $\chi(M) = 2$. The induction assumption is that all $(m-1)$-spheres S satisfy $\chi(S) = 1 + (-1)^{m-1}$. Pick a vertex v. As the unit sphere $S(v)$ is a $(m-1)$-sphere and $S(v) = B(v) \cap G \setminus v$, where both the unit ball $B(v)$ and $G \setminus v$ are contractible with Euler characteristic 1, we have, using the induction assumption, $\chi(M) = \chi(G \setminus v) + \chi(B(v)) - \chi(G \setminus v \cap B(v)) = 2 - (1 - (-1)^{m-1}) = 1 + (-1)^m$. □

19.4. In the continuum, manifolds can be constructed as level surfaces of functions like $x^2 + y^2 + z^2 = 1$. We can do that also in the discrete. Take an arbitrary function $f : V \to Z_k = \{0, \ldots, k\}$. It defines a new graph M_f, where the vertices are the set of complete subgraphs on which f attains all k values. Connect two of these points by an edge, if one is contained in the other. The new graph M_f is a sub-graph of the **Barycentric refinement** of M. There is the analog of what we have seen classically for functions on manifolds.

Theorem 2 (Level Sets)**.** *If M is a m-manifold and $f : M \to Z_k$ is an arbitrary function, then either M_f is empty or then M_f is a $(m-k)$-manifold.*

Proof. Let x be a n-simplex on which f takes all values. This means $f(x) = Z_k$. The graph $S^-(x) = \{y \subset x, y \neq x\}$ is a $(n-1)$-sphere in the Barycentric refinement of M. The simplices in $S^-(x)$ on which f still reaches Z_k is by induction a $(n-1-k)$-manifold and since we are in a simplex, it has to be a $(n-1-k)$-sphere. Every unit sphere $S(x)$ in the Barycentric refinement is a $(m-1)$-sphere as it is the join of $S^-(x)$ with $S^+(x) = \{y, x \subset y, x \neq y\}$. (Pset 9: The join of two spheres is always a sphere.) The sphere $S_f^+(x)$ in M_f is the same than $S^+(x)$ in M because every simplex z in M containing x automatically has the property that $f(z) = Z_k$. So, the unit sphere $S(x)$ in M_f is the join of a $(n-k-1)$-sphere and the $(m-n-1)$-sphere and so a $(m-k-1)$-sphere. Having shown that every unit sphere in M_f is a $(m-k-1)$-sphere, we see that M_f is a $(m-k)$-manifold. □

19.5. Differential geometry works too: define **curvature** as

$$K(v) = \sum_{k=0}^{m} \frac{(-1)^k f_{k-1}(S(v))}{k+1} = 1 - \frac{f_0(S(v))}{2} - \frac{f_1(S(v)))}{3} + \ldots \; .$$

In the case of a 2-manifold, this boils down to $1 - f_0(S(v))/2 + f_1(S(v))/3 = 1 - d(v)/6$, where $d(v)$ is the vertex degree. For odd-dimensional manifolds, the curvature is constant zero. Note that the following holds for all graphs!

Theorem 3 (General Gauss-Bonnet)**.** $\sum_{v \in V} K(v) = \chi(M)$

Proof. The proof is the same as in the 2-dimensional case. Again look at the energies $\omega(x) = (-1)^{\dim(x)}$ attached to each simplex x in the graph (complete subgraph with $\dim(x) + 1$ vertices). Then $\chi(M) = \sum_x \omega(x)$. Now distribute all these energies of a k-simplex x equally to the $k+1$ vertices contained in x. As there are f_{k-1} simplices in $S(v)$ which correspond do simplices containing v, this adds $\frac{(-1)^k f_{k-1}(S(v))}{k+1}$ to each vertex v. Now just collect all up at a vertex v to get the curvature $K(v)$. The transactions of energies preserved the total energy = Euler characteristic. □