DIFFERENTIAL GEOMETRY

MATH 136

Unit 18: Riemannian Manifolds

18.1. A **topological manifold** M **of dimension** m is a metric space (i.e. a subset of some $\mathbb{R}^n$) such that every $x \in M$ has a neighborhood U, that is homeomorphic to an open subset $R = \phi(U)$ of $\mathbb{R}^m$. The pair (U, ϕ) is called a **chart**. It produces a **coordinate system** on U: there is a parametrization $r(\phi(x)) = x$, which is a regular map from $R \subset \mathbb{R}^m$ to $\mathbb{R}^n$, meaning that dr has rank m everywhere. A C^k **atlas** on M is a collection $\mathcal{F} = \{U_i, \phi_i\}_{i \in I}$ of charts such that $\bigcup_{i \in I} U_i = M$, and that all coordinate change maps $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are in $C^k(\phi_i(U_j \cap U_i), \mathbb{R}^m)$. An atlas is called **maximal**, if (U, ϕ) is a chart such that $\phi \circ \phi_i^{-1}$ and $\phi_i \circ \phi^{-1}$ are all C^k for all $i \in I$, then $(U, \phi) \in \mathcal{F}$. Two atlases $\mathcal{F}, \mathcal{G}$ are called **equivalent** if their union $\mathcal{F} \cup \mathcal{G}$ is an atlas. Given an atlas $\mathcal{F}$, the union of all atlases equivalent to $\mathcal{F}$ is called a **differentiable structure of** $\mathcal{F}$. [1]

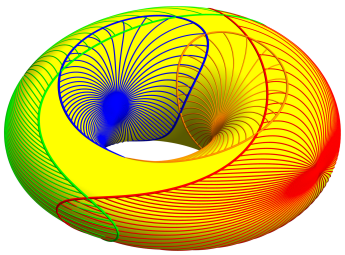


FIGURE 1. A m-manifold $M \subset \mathbb{R}^n$ is shown with part of an atlas $\mathcal{F}$. Each patch U_i is a regularly parametrized by $r : R_i \to U_i$ with $R_i = \phi_i(U_i) \subset \mathbb{R}^m$. The map r has maximal rank m everywhere on R_i.

18.2. A m-dimensional C^k**-differentiable manifold** is a pair $(M, \mathcal{F})$, where M is a m-dimensional topological manifold and $\mathcal{F}$ is a differentiable C^k structure on M. This means is that every point $p \in M$ is contained in a region $U = \phi(R)$. The inverse r of ϕ defines a parametrization from $R \subset \mathbb{R}^m$ to $M \subset \mathbb{R}^n$. We are now in the frame

[1] The concept can be difficult: $\mathcal{F}$ is not unique in general. On S^7, there are 28 different smooth structures. The smooth Poincare conjecture claims that S^4 has a unique differentiable structure.

work used before. We can for example define r_u, r_v and the fundamental forms. We say **smooth** if we just want to be able to differentiate as often as necessary. [2]

18.3. If $E = \mathbb{R}^m$ is the space of **column vectors** of dimension m, its **dual** E^* is defined as the space of all linear maps $f : E \to \mathbb{R}$, It is the space of **row vectors**. If $\{e_1, \dots, e_m\}$ is a basis of E, then $\{e^1, \dots, e^m\}$ denotes a basis of E^*. Every element in E can be written as $v = \sum_i v^i e_i$, every element in E^* can be written as $v = \sum_i v_i e^i$. For $p, q \geq 0$, the linear space T^p_q of all multi-linear maps $(E^*)^p \times E^q \to \mathbb{R}$ is called the space of **tensors of type** (p, q). Column vectors are $(1, 0)$-tensors in $T^1_0 = E$. Row vectors are $(0, 1)$-tensors in $T^0_1 = E^*$, bilinear maps are $(0, 2)$-tensors in T^2_0. A **tensor field of type** (p, q) on a m-manifold M is a smooth assignment of a (p, q) tensor to every point. Such a map is also called a **section** of the tensor bundle, generalizing that a **vector field**. A vector field is a section of the **tangent bundle** TM. For a $(0, 2)$ tensor field g for example, the attribute "smooth" means that for any vector fields X, Y, the function $x \to g(x)(X(x), Y(x))$ is smooth from M to $\mathbb{R}$. If $f : M \to \mathbb{R}^k$ is a smooth map, then df is a $(0, 1)$ tensor field. This is also called a **1-form**. **Vector field** is an abbreviation for a $(1, 0)$ tensor field. The first fundamental form g is by definition a $(0, 2)$ tensor field; think of a symmetric bilinear form attached to every point. A **Riemannian manifold** (M, g) is a smooth manifold M with a positive definite symmetric $(2, 0)$ tensor field g.

18.4. Let M be a m-manifold and $f : M \to \mathbb{R}^k$ be smooth. A point $x \in M$ is called a **critical point** and $f(x)$ a **critical value**, if the rank of $df(x)$ is not m. Non-critical points are called **regular points**.

Theorem 1. *If M is a m-manifold and $f : M \to \mathbb{R}^k$ is smooth and y is a regular value of f, then $M_f = f^{-1}(y)$ is a manifold of dimension $m - k$.*

Proof. At $x \in f^{-1}(y)$, the Jacobean map $df(x)$ has rank k and the kernel $H = \ker(df)$ of $df(x)$ is $(m - k)$-dimensional and $H^\perp$ is k dimensional. Take a chart (U, ϕ) in M which contains x. It defines a parametrization $r = \phi^{-1} : R \to U$. Define $g = f \circ r : R \subset \mathbb{R}^m \to \mathbb{R}^k$. Define the projection $L : \mathbb{R}^m = H \oplus H^\perp \to \mathbb{R}^{m-k}, (h, h') \mapsto h'$ onto the orthogonal complement of the kernel. The map $F : R \subset \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ defined by $F(z) = (g(z), Lz)$ has the non-singular derivative $dF(u) = (dg(u), Lu)$. By the **inverse function theorem**, a neighborhood of $u = \phi(x)$ is mapped by the diffeomorphism F onto a neighborhood $F(y)$ of $F(u)$. We get so a chart $U_x = r(R_x)$, where $R_x = F^{-1}(H)$. It is a chart of M_f. Doing the same construction at any point $x \in M$ produces an atlas for $f^{-1}(y)$ and verifies that $f^{-1}(y)$ is a $(m-k)$-manifold. □

18.5. Examples: a) The **d − sphere** is the set $M = S^d = \{x \in \mathbb{R}^{d+1} \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\} \subset \mathbb{R}^{d+1}$. The **standard differentiable structure** $\mathcal{F}$ on S^d is generated by $\mathcal{F} = \{(S^d \setminus \{A\}, \phi_A), (S^d \setminus \{B\}, \phi_B)\}$, where $\phi_A \phi_B$ are **stereographic projections** from antipodal points A, B.
b) The set $SL(n, \mathbb{R})$ of $n \times n$ matrices of determinant 1 is a manifold.

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[2] A theorem of Whitney assures that a smooth compact m-manifold M given abstractly as paracompact Hausdorff space can be realized within $\mathbb{R}^n$ if $n = 2m + 1$.