DIFFERENTIAL GEOMETRY

MATH 136

Unit 17: Global Gauss-Bonnet

17.1. Now we are ready to prove the **global Gauss-Bonnet theorem** for a 2-manifold M without boundary. The surface M is triangulated by a **discrete manifold** $G = (V, E, F)$, where the faces F are the triangles defined by the graph (V, E). The discrete manifold is geometrically realized in M as a collection of points, a collection of curves connecting vertices. The geometrically realized network divides M up into triangular faces $M_i = r(U_i)$. The Euler characteristic of M is $\chi(M) = V - E + F$. As we have seen, $\chi(M)$ does not depend on the triangulation: topological changes like removing a disc and gluing in a new disc (we called this as a **connected sum** $M \to M \# S^2$) or doing a **Barycentric refinement** does not change $\chi(M)$.

Theorem 1 (Gauss-Bonnet theorem)**.** *For a compact* 2*-manifold,* $\iint_M K dV = 2\pi\chi(M)$.

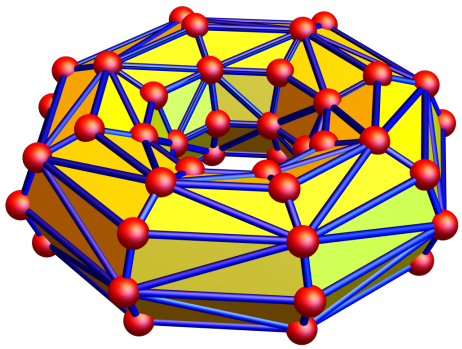


FIGURE 1. A triangulated manifold M. We apply the local Gauss-Bonnet theorem on each of the triangles. The edge contributions produce with Green's theorem the face curvatures $\iint_{M_i} K \; dV$ as well as $2\pi F$. The vertex contributions produce, using the Euler Handshake lemma, $2\pi(V - E)$. Overall, we have $2\pi(V - E + F) = 2\pi\chi(M)$. The picture shows a $M = \mathbb{T}^2$ with $V = 64$ vertices.

17.2. We will use the local Gauss-Bonnet theorem for each triangle U_i with angles $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$. We first of all want to understand what happens if we glue together triangles such that the frame field $X = [zw_u, zw_v]$ can be define on the union. On the curve obtained by intersecting two adjacent triangles, the line integral of X cancels.

Lemma 1 (Cancellation)**.** *If two triangles M_1, M_2 meet in a curve C and C_1, C_2 are the parametrizations matching the M_1, M_2, then $\int_{C_1} X\, dr + \int_{C_2} X\, dr = 0$.*

Proof. If X is a 1-form and C is a curve and $-C$ is the curve passed backwards, then $\int_C X\, dr + \int_{-C} X\, dr$. This is what happens here. You can see the identity also as a consequence of Green also, noting that $C \cup -C$ encloses an "empty region". □

17.3. We see that all the 1-form contributions from the edges are zero. The contributions from the faces M_i add up:

Lemma 2 (Additivity)**.** $\int_M K\, dV = \sum_i \int_{M_i} K\, dV$.

Proof. The patches M_i are all disjoint. Their union is $\bigcup_i M_i = M$. Areas of disjoint regions add up. □

17.4. The contributions from the vertex degrees $d_i = |S(v_i)|$ add up too. [1]

Lemma 3 (Euler handshake)**.** *If (V, E) has vertex degrees d_i, then $2E = \sum_i d_i$.*

Proof. You prove this in a homework. □

17.5. We still have to look at the contributions from the vertices. At each point P_i we have angles α_{ij} for $j = 1, \ldots, d_j$, where d_j is the vertex degree.

Lemma 4 (Adding vertex curvatures)**.** $\sum_{i=1}^{V} \sum_{j=1}^{d_i} \kappa_{ij} = 2\pi E - 2\pi V$.

Proof. (i) $\sum_{i=1}^{V} \sum_{j=1}^{d_i} \kappa_{ij} = \sum_{i=1}^{F} \sum_{j=1}^{3} (\pi - \alpha_{ij})$. (ii) $\sum_{k=1}^{V} \sum_{j=1}^{d_k} \pi = 2\pi E$. (iii) $\sum_{k=1}^{V} \sum_{j=1}^{d_k} \alpha_{kj} = 2\pi V$. □

[2]

17.6. Proof of the global Gauss-Bonnet theorem:
The local Gauss-Bonnet theorem told us $\sum_i [\iint_{U_i} K dV + \sum_j \kappa_{ij} - 2\pi] = 0$. This means that $\iint_U K\, dV + \sum_{i,j} \kappa_{ij} - 2\pi F = 0$. Therefore, using the previous lemma:

$$\iint_U K\, dV = 2\pi V - 2\pi E + 2\pi F = 2\pi \chi(M) \; .$$

17.7. 1) If M is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, then $\iint_M K\, dV = 4\pi$ (HW).
2) A genus k surface has $\iint_M K\, dV = 2\pi(2 - 2k)$. For a torus $\iint_M K\, dV = 0$.
3) A Klein bottle is obtained by gluing two Möbius strips together. $\iint_M K\, dV = 0$. Because each Möbius strip has Euler characteristic 0 (you computed that in an example), and the Möbius strip can be realized so that the boundary curvature χ_g is zero.

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[1] If V, E, F are the vertices, edges and faces. It is custom to write its cardinalities as V, E, F.
[2] See youtube video with $3F = 2E$ as in the lecture.