

DIFFERENTIAL GEOMETRY

MATH 136

Unit 12: Geodesics

12.1. If $M = r(R)$ is a regular manifold, define the space $\mathcal{X}$ of regular paths $x(t)$ that start at $A = x(a) \in R$ and end at $B = x(b) \in R$. If $F(x, \dot{x})$ is a C^1 function of position x and velocity $\dot{x}$, we can minimize $E(x) = \int_a^b F(x, \dot{x}) dt$ by looking for paths $x(t)$ at which the variation is zero.¹

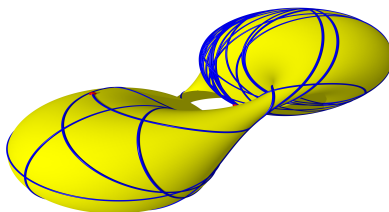


FIGURE 1. This geodesic on a torus was computed in Povray with the **Schild's ladder method**: evolve freely in $\mathbb{R}^3$ but stay glued to the surface.

Theorem 1 (Euler-Lagrange). *If x minimizes E , then*
$$\boxed{F_x(x, \dot{x}) = \frac{d}{dt} F_{\dot{x}}(x, \dot{x})}.$$

Proof. For a minimum, the change $E(x + \xi) - E(x)$ of a displacement $x + \xi$ of x satisfies $\int_a^b F(x + \xi, \dot{x} + \dot{\xi}) - F(x, \dot{x}) dt \geq 0$. As Fermat knew, we better have $dE\xi = \lim_{h \rightarrow 0} (E(x + h\xi) - E(x))/h = 0$ because a non-zero limit would make $E(x + h\xi)$ larger or smaller than $E(x)$ for small h . By the chain rule, $dE\xi = \int_a^b F_x(x, \dot{x})\xi + F_{\dot{x}}(x, \dot{x})\dot{\xi} dt$. Integration by parts, using $\xi(a) = \xi(b) = 0$, gives $dE\xi = \int_a^b [F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x})]\xi(t) dt$. In order that this is zero for all ξ , we better have $[F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x})] = 0$ for all $t \in [a, b]$. Proof. If $\neq 0$ at some point $t \in [a, b]$, it would be non-zero in a neighborhood U of t , allowing to find a smooth function ξ that is positive in U and 0 else, producing a nonzero change $dE\xi$. $\square$

¹Using $x(t) = (x^1(t), \dots, x^m(t)) = (u^1(t), \dots, u^m(t))$ is traditional. For typographical reasons, write $\dot{x}^k$ rather than x'^k . If $r(u, v)$ parametrizes M , paths $x(t) = (u(t), v(t)) \in R$ define curves $r(x(t)) \in M$. “Variation” instead of “derivative” avoids confusion with $\dot{x}$. Variations are directional derivatives in an infinite dimensional space $\mathbb{X}$ of paths between two fixed points.

12.2. To understand minima if $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle = g(x)(\dot{x}, \dot{x}) = \sum_{i,j} g_{ij}(x) \dot{x}^i \dot{x}^j$, we need notation. M was defined as a regular map $r : R \rightarrow \mathbb{R}^n$ giving points $r(u^1, \dots, u^m) \in \mathbb{R}^n$. Define the **Christoffel symbols** $\boxed{\Gamma_{ijk} = r_{u^i u^j} \cdot r_{u^k}}$. The product rule gives

$$\partial_{u^k} g_{ij} = r_{u^i u^k} \cdot r_{u^j} + r_{u^i} \cdot r_{u^j u^k} = \Gamma_{ikj} + \Gamma_{jki} ,$$

$$\partial_{u^i} g_{jk} = r_{u^j u^i} \cdot r_{u^k} + r_{u^j} \cdot r_{u^k u^i} = \Gamma_{jik} + \Gamma_{kij} ,$$

$$\partial_{u^j} g_{ki} = r_{u^k u^j} \cdot r_{u^i} + r_{u^k} \cdot r_{u^i u^j} = \Gamma_{kji} + \Gamma_{ijk} .$$

Adding the second and third and subtracting the first, using Clairaut $\Gamma_{ijk} = \Gamma_{jik}$, gives $2\Gamma_{ijk}$ on the right hand side. So:

Lemma 1.

$$\boxed{\Gamma_{ijk} = \frac{1}{2} \left[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij} \right] .}$$

Using the notation $g^{ij} = (g^{-1})_{ij}$ and $\Gamma_{ij}^k = \sum_{l=1}^m g^{kl} \Gamma_{ijl}$, we get to the main point: ²

Theorem 2 (Geodesics). *Minima of the action functional $E(x) = \int_a^b \langle \dot{x}, \dot{x} \rangle dt$ satisfy*

$$\boxed{\ddot{x}^k + \sum_{i,j=1}^m \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0.}$$

Proof. Goal: Euler-Lagrange for $F(x, \dot{x}) = \sum_{i,j} g_{ij}(x) \dot{x}^i \dot{x}^j$ is $2 \sum_j g_{jk} \ddot{x}^j + 2 \sum_{i,j} \Gamma_{ijk} \dot{x}^i \dot{x}^j = 0$; use notation $\partial_{x^k} g_{ij} = g_{ij,k}$. First get $F_{\dot{x}^k} = \sum_j g_{kj} \dot{x}^j + \sum_j g_{jk} \dot{x}^j$. Then $\frac{d}{dt} F_{\dot{x}^k} - F_{x^k} = \sum_{j,i} g_{kj,i} \dot{x}^i \dot{x}^j + \sum_j g_{kj} \ddot{x}^j + \sum_{j,i} g_{jk,i} \dot{x}^i \dot{x}^j + \sum_j g_{jk} \ddot{x}^j - \sum_{i,j} g_{ij,k} \dot{x}^i \dot{x}^j$. The 1st, 3rd and 5th terms add up to $2 \sum_{i,j} \Gamma_{ijk} \dot{x}^i \dot{x}^j$. The 2nd and 4th give $2 \sum_j g_{jk} \ddot{x}^j$. $\square$

12.3. Acceleration of a particle moving on a geodesic is determined by the velocity and a “gravitational force” terms Γ which involves change of the metric which Einstein would identify with “energy” or “mass”.

12.4. With $G(x, \dot{x}) = \sqrt{g_{ij}(x) \dot{x}^i \dot{x}^j} = \sqrt{F}$, we get the **arc length functional**

$$I(r) = \int_a^b \|\dot{x}\| dt = \int_a^b \sqrt{\langle \dot{x}(t), \dot{x}(t) \rangle} dt = \int_a^b \sqrt{\sum_{i,j} g_{ij}(x) \dot{x}^i \dot{x}^j} dt .$$

Theorem 3 (Maupertius). *Action and length functionals have the same extrema.*

Proof. Because $\frac{d}{dt} G_{\dot{x}} = \frac{F_{\dot{x}}}{2\sqrt{F}}$ and $G_x = \frac{F_x}{2\sqrt{F}}$, the Euler-Lagrange equations $\frac{d}{dt} G_{\dot{x}} = G_x$ are equivalent to the Euler-Lagrange equations $\frac{d}{dt} F_{\dot{x}} / \sqrt{F} = F_x / \sqrt{F}$. We have used that x was regular, so that $F(x, \dot{x})$ is never zero. $\square$

12.5. If $x(t)$ arc length parametrizes a curve on M , the **normal curvature** is defined as $\boxed{\kappa_n = \ddot{x} \cdot n}$. It is the scalar projection acceleration $\ddot{x}$ onto n . By Cauchy-Schwarz, it is smaller or equal than $\kappa = |\ddot{x}|$. In 3D, the **geodesic curvature** is $\kappa_g = \ddot{x} \cdot (n \times \dot{x})$, in general $\kappa_g = |P(\ddot{x})| = |\ddot{x} - (\ddot{x} \cdot n)n|$. Pythagoras gives $\kappa_n^2 + \kappa_g^2 = \kappa^2$. ³

Theorem 4 (Schield’s ladder). *Geodesics have zero geodesic curvature.*

This means that the projection of $\dot{x}$ onto the tangent space is zero. To evolve the geodesic, we can now evolve freely in the ambient space for a short time, then project to the surface. This means that the body moves freely but gets attracted to the surface.

²It’s musical! ∂_{u^i} is co-variant and u^i is contra-variant. Einstein would write $g_{ij} \dot{x}^i \dot{x}^j$ for $\langle \dot{x}, \dot{x} \rangle$.

³ κ_n is signed in general. In 3D, κ_g is signed.