

DIFFERENTIAL GEOMETRY

MATH 136

Unit 8: Four vertex theorem

8.1. A **vertex** of a C^2 planar curve is a point $r(t)$ where $\kappa(t)$ has a maximum or minimum. For a non-circular ellipse for example, there are two maxima and two minima, and so 4 vertices in total. What happens in general? A curve is called **convex** if it bounds a convex region R . A region $R \subset \mathbb{R}^2$ is called **convex** if the line segment $[A, B]$ between any two points $A, B \in R$ is part of the region.

Theorem 1. *A simple closed regular convex C^3 plane curve C has at least 4 vertices.*

Proof. We can assume κ is not constant and that C is parametrized by arc length. A continuous function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has by the extremal value theorem at least one maximum a and one minimum b . For finitely critical points on the circle, the number of critical points is even because they alternate. Assume, $r(a)$ and $r(b)$ are the only critical points and that $r(a)$ is the maximum and $r(b)$ the minimum. We prove that there exists one more critical point. Choose the coordinate system so that $r(a), r(b)$ are on the x -axis so that $r(t) = [x(t), y(t)]$ and $y(a) = y(b) = 0$. Convexity assures that there is no other root $y(t) = 0$. So, k' changes sign at $t = a$ and $t = b$ and nowhere else and $k'(t)y(t)$ does not change sign at all. The Frenet equations tell $T = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $N = \begin{bmatrix} -y' \\ x' \end{bmatrix}$ satisfy $T' = \kappa N$. The first coordinate gives $x'' = -\kappa y'$. Integrate over the entire curve gives $\int_0^L \kappa'(t)y(t) dt = \kappa y|_0^L - \int_0^L \kappa(t)y'(t) dt = \int_0^L x''(t) dt = x'(L) - x'(0) = 0$ because C is closed. If there would be no further critical point of κ , then $g(t) = \kappa'(t)y(t)$ was non-negative on $[a, b]$ as $\kappa'(t) \leq 0, y(t) \leq 0$. Similarly, on $[b, L + a]$ we would have $\kappa'(t) \geq 0$ and $y(t) \geq 0$. Having $g(t) \geq 0$ and $\int_0^L g(t) dt = 0$ forces $g(t) = 0$ for all t so that $\kappa'(t) = 0$ contradicting we assumed κ to be non-constant. The contradiction proves there exists at least one more critical point and so 4. $\square$

8.2. Examples. 1) The ellipse $r(t) = [2\cos(t), \sin(t)]$ has the curvature $\kappa(t) = 2(\cos^2(t) + 4\sin^2(t))^{-3/2}$ which has maxima at $t = 0, \pi$ and minima at $t = \pm\pi/2$.
2) The curve $r(t) = 5[\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature $r'(t) \times r''(t) / |r'(t)|^{3/2}$ that has minima at $0, \pi$ and maxima at $\pm \arccos(-2/5)$.
3) The **limaçon** $r(t) = [\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature with only 2 vertices! Why is this not a counter example?

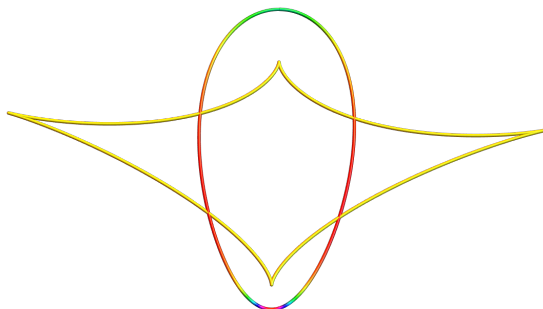


FIGURE 1. A simple closed curve for which curvature is coded as color. The 4 vertices are visualized by plotting the **evolute** $e(t) = r(t) + n(t)/\kappa(t)$, where $n(t)$ is the normal vector pointing inside. The vertices of the curve correspond to cusps of the evolute.

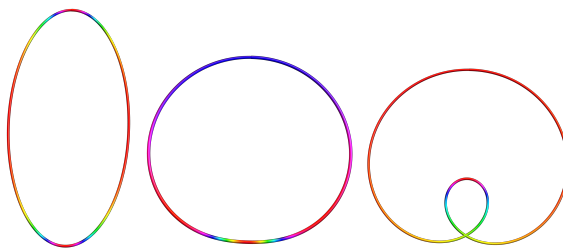


FIGURE 2. The ellipse, a deformation of an ellipse and a limaçon.

8.3. Remarks. 1) Convexity is not really needed. Osserman showed that for any simple closed C^2 curve C there are $2n$ vertices if the smallest circle enclosing C intersects it in at least n connected components. Also, a general simple closed C^2 curve has at least 4 vertices.

2) V. Arnold conjectured that the result holds for any curve that can be obtained from a circle by suitable deformations.

3) Converse result: If a continuous real valued, periodic function has at least two local maxima and two local minima, it is the curvature function of a simple closed curve.

8.4. History. 1) The theorem was proven in 1909 by Syamadas Mukhopadhyaya for convex curves. 2) The general case was published in 1912 by Adolph Kneser. 3) The proof given here in the convex case is due to G. Herglotz in 1930. 4) Robert Osserman in 1985 generalized the result to the "four or more vertex theorem". 5) The converse result started with H. Gluck in 1971 and was proven in 1997 by Bjoern Dahlberg.

8.5. Related. 1) The **evolute** of a plane curve is defined as $e(t) = r(t) + n(t)/\kappa(t)$. It is the caustic of the normal map. At points, where $\kappa'(t)$ is zero, the evolute has **cusps**. A caustic of a simple closed curve therefore has at least 4 cusps. For the ellipse the evolute is called the Lamé curve. 2) The **tennis ball theorem** states that a C^2 curve on the sphere that divides the sphere into regions of equal area have at least 4 inflection points. 3) The open **last geometrical problem of Jacobi** asks whether a caustic on an ellipse has at least 4 cusps.