

# DIFFERENTIAL GEOMETRY

MATH 136

## Unit 5: Frenet Equations

**5.1.** Today, we work at smooth curves  $r : [a, b] \rightarrow \mathbb{R}^3$ . We will require the map  $r$  to be at least three times continuously differentiable and say  $r$  is a  $C^3$  **curve**. Define the **unit tangent vector**  $T(t) = r'(t)/|r'(t)|$ , the **normal vector**  $N(t) = T'(t)/|T'(t)|$  and the **binormal vector**  $B(t) = T(t) \times N(t)$ . The three vectors are defined, as long as  $r'$  and  $T'$  are both non-zero. One calls it a **Frenet frame**  $(T, N, B)$ . A  $C^3$  curve is called a **Frenet curve** if  $r', r''$  are linearly independent at every  $t$ . This is equivalent to the statement  $\boxed{r' \times r'' \neq 0}$  for every  $t$ .

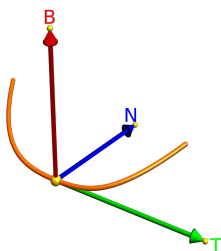


FIGURE 1. Direction  $T$ , normal  $N$  and binormal direction  $B$ .

**Theorem 1.** *For a Frenet curve, the Frenet frame  $TNB$  is orthonormal at every point.*

*Proof.* By assumption,  $r'$  and  $r''$  are both not zero. If the parametrization is arc length, then  $T = r'$  and  $N = T'/|T'| = r''/|r''|$ . Since  $T \cdot T = 1$ , we have by Leibniz product rule,  $2T' \cdot T = 0$  so that  $N$  is perpendicular to  $T$ . The cross product  $T \times B$  now also has length 1 and is perpendicular to both  $T$  and  $N$ .  $\square$

**5.2.** Every Frenet curve  $r(t)$  can be parametrized by arc length as you prove yourself. The **curvature**  $\kappa$  is then defined as  $\boxed{\kappa = T' \cdot N}$  which is  $|T'|$ . The curvature measures the deviation of the curve from being linear. The **torsion**  $\tau$  is defined as  $\boxed{\tau = N' \cdot B}$ . It measures the deviation from the curve of being planar. We can encode the three vectors  $T, N, B$  by turning them into row vectors of an **orthogonal**  $3 \times 3$  **matrix**  $Q(t) = \begin{bmatrix} T & N & B \end{bmatrix}^T$ . We get now  $Q'(t) = K(t)Q(t)$ , where  $K(t)$  is skew-symmetric:

## 5.3.

**Theorem 2** (Frenet equations). 
$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

*Proof.* Because  $N \cdot T = 0$ , we have  $N' \cdot T = -T' \cdot N = -\kappa$ . The relation  $N \cdot N = 1$  implies  $N' \cdot N = 0$ . The relation  $B \cdot B = 1$  implies  $B' \cdot B = 0$ . Now expand with respect to the basis  $\{T, N, B\}$  and use  $\kappa = |T'|$  and  $\tau = N' \cdot B$ :

$$\begin{aligned} T' &= (T' \cdot T)T + (T' \cdot N)N + (T' \cdot B)B = 0 + 0 + \kappa N \\ N' &= (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B = -\kappa T + 0 + \tau B \\ B' &= (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B = 0 - \tau N + 0. \end{aligned}$$

□

**5.4.** In  $\mathbb{R}^2$ , we only have  $T$  and  $N$ . We can reduce to the planar case if  $\tau$  is constant 0. The Frenet equations are then 
$$\begin{bmatrix} T \\ N \end{bmatrix}' = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}.$$

**5.5.** The **fundamental theory of curves** in  $\mathbb{R}^3$  tells that curvature and torsion determines a curve up to Euclidean congruences given by rotations or translations.

"The shape of a Frenet curve is determined by curvature and torsion".

**Lemma 1.** *For any differentiable curvature and torsion functions  $\kappa(t) \geq 0$  and  $\tau(t)$ , there exists up to translation and rotation a **unique** curve  $r(t)$  parametrized by arc length that has the given curvature and torsion.*

*Proof.* The initial  $r(0)$  and  $(T(0), N(0), B(0))$  fixes the initial point and location. Now "build the curve": the functions  $\kappa(t), \tau(t)$  define a **skew symmetric** matrix

$$K(t) = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}. \text{ We define orthogonal matrix } Q(t) \text{ satisfying the differential}$$

equation  $Q'(t) = K(t)Q(t)$ . In the homework you will verify that a differential equation  $x' = F(t, x)$  with a  $C^1$  function  $F$  has locally a unique solution and that if  $x(t)$  stays bounded, the solutions exist for all times. You also will check that if  $Q(t)$  is a curve of orthogonal matrices, then  $Q' = KQ$  with skew symmetric  $K$ . This can be reversed: if  $Q(0)$  is orthogonal and  $K(t)$  is skew symmetric, then the solution  $Q(t)$  of the differential equation is orthogonal. Having  $Q(t)$ , it produces  $r'(s) = Q(s)r'(0)$  and so  $r(t) = r(0) + \int_0^t r'(s) ds$ , where  $r'(t) = r'(0) + \int_0^t Q^T(s)r'(0) ds$ . □

**5.6.** You might have seen in calculus courses formulas for curvature and torsion for a curve  $r(t)$ . These formulas worked if the curve was not necessarily arc-length parametrized. In the Frenet case, meaning that  $r' \times r'' \neq 0$ , we will prove them in class:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}, \quad \tau(t) = \frac{\det[r'(t), r''(t), r'''(t)]}{|r' \times r''|^2}.$$

In class we will look at examples where we can compute the curvatures and torsion. A case where  $\kappa$  is arc length and a case where  $\kappa = \tau$ .