

DIFFERENTIAL GEOMETRY

MATH 136

Unit 2: Preliminaries

2.1. A map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \mapsto \begin{bmatrix} f_1(x_1, \dots, x_m) \\ \vdots \\ f_n(x_1, \dots, x_m) \end{bmatrix}$ is called **differentiable** or C^1 , if all derivatives $\frac{\partial}{\partial x^j} f_i$ are continuous. For such a map, define the $n \times m$ **Jacobian matrix** $df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_1(x_1, \dots, x_m) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} f_n(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_n(x_1, \dots, x_m) \end{bmatrix}$.

- 2.2.** a) If $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a curve, then $dr(t) = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ is the **velocity**.
 b) If $f(x, y, z)$ is a function of 3 variables, then $df(x, y, z) = [f_x, f_y, f_z]$ is a 1×3 matrix. It is the transpose of the gradient.
 c) A vector field $f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$ has $df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ is called the Jacobian.
 d) If $f(x) = Ax$ is a linear map, then $df(x) = A$.

2.3. We say df has **maximal rank** if its rank is the minimum of m, n . If $m < n$, then the image of f is in general a m -dimensional set in $\mathbb{R}^n$. If $n < m$, then we can look at the **roots** $\{f(x) = 0\}$ which is in general a $(n - m)$ -dimensional set. If $m \leq n$, we can say that f is **manifold like near** $f(x)$ if $df(x)$ has maximal rank. If $n \leq m$, then $M = \{f - c = 0\}$ is **manifold like near** x , if $df(x)$ has maximal rank. The **fundamental theorem of linear algebra** is $\ker(A^T) = \text{im}(A)^\perp$.

2.4. Example: $m = 2, n = 1$ and $f(x, y) = x^3y^2 - xy^3 = -4$. What happens near $(x_0, y_0) = (1, 2)$. To investigate this, we look at the Jacobean matrix $df(x, y) = [3x^2y^2 - y^3, 2x^3y - 3xy^2]$ which is at $(1, 2)$ equal to $[4, -8]$. The line $4x - 8y = -12$ is tangent to the curve. If we write the curve as $y = g(x)$ near $x = 1$, then $f(x, g(x)) = -4$ and differentiating gives $f_x + f_y g' = 0$ so that $g' = -f_x/f_y$. It exists:

Theorem 1 (Implicit function theorem). *If $f(x, y) = f(x_0, y_0) = 0$ and $f_y(x_0, y_0) \neq 0$, then $f(x, y) = 0$ can near (x_0, y_0) be written as $y = g(x)$ for some C^1 function $g(x)$.*

Proof. Take a small neighborhood $U = [x_0 - a, x_0 + a] \times [y_0 - a, y_0 + a]$ of (x_0, y_0) . such that $|f_y(x, y)| \geq c$ and $|f_x(x, y)| \leq d$ for all $(x, y) \in U$. Choose $a > 0$ so small that $f(x, y_0 - a)f(x, y_0 + a) < 0$ for all $x \in [x_0 - a, x_0 + a]$. By the **intermediate value theorem**, there is for every $x \in [x_0 - a, x_0 + a]$ a value $g(x)$ such that $f(x, g(x)) = 0$. The curve $f(x, y) = 0$ agrees with the curve $y = g(x)$ in the window U . To see

that g is differentiable, use the chain rule $\frac{\partial}{\partial x} f(x, g(x)) = f_x \cdot 1 + f_y \cdot g' = 0$, so that $g'(x) = -f_x(x, g(x))/f_y(x, g(x))$ and $|g'(x)| \leq d/c$ is bounded in U . $\square$

2.5. It follows that if the $df(x_0, y_0) \neq [0, 0]$, then then the contour set $f(x, y) = c$ is near (x_0, y_0) the graph of a function: Proof: if $f_y \neq 0$ use the implicit function theorem giving $y = g(x)$. If $f_y = 0$, and so $f_x \neq 0$ we can write $x = g(y)$ for some C^1 function g .

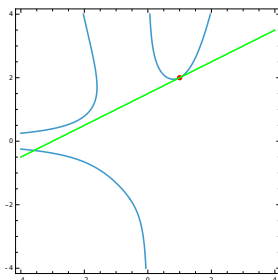


FIGURE 1. $f(x, y) = 0$ can near (x_0, y_0) be written as a graph $y = g(x)$.

2.6. If $m < n$, then f has maximal rank if df has rank m . We look at this case more next week and look at $g = df^T df$. Lets look at a C^1 curve C defined by the parametrization $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Its velocity is $dr = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$. If the velocity is not zero at $t = t_0$ then the curve is manifold like. The curve is then close to the line $l(s) = r(t_0) + sr'(t_0)$. If $r'(t_0) \neq 0$, then C is a graph $(s, g(s))$ close to the line $(s, l(s))$.

2.7. In the case $m = n$, the map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the image of f is manifold like near x if $\det(df) \neq 0$. In that case f is invertible near x :

Theorem 2 (Inverse function theorem). *If $y \mapsto h(y) \in C^1$ and $h'(y_0) \neq 0$, then there exists $g \in C^1$ such that $g(h(x)) = 0$ with $dg(x_0) = dh(y_0)^{-1}$.*

Proof. Define $f(x, y) = x - h(y) = 0$. As $f_y = -h'(y)$ is non-zero, the implicit function theorem applies: there is $g(x)$ with $y = g(x)$ near x_0 . For $y = g(x)$ in the range of g define $h(y) = x$. This is the inverse we were looking for. $\square$

2.8. All proofs can be generalized to maps $f : \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x, y) \mapsto f(x, y) \in \mathbb{R}^n$. Now f_x is a $n \times k$ matrix and f_y is a $n \times n$ matrix. If f_y is invertible, then df has maximal rank and $f(x, y) = c$ can be written as $y = g(x)$ and $n \times k$ matrix $dg = -f_y^{-1} f_x$. This is the implicit function theorem. If $k = 0$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, then $df(x)$ invertible implies f is invertible near x .

2.9. A point $x \in \mathbb{R}^m$ is a **critical point** of f if $df(x)$ does not have maximal rank. The value $f(x)$ is then called a **critical value**. Sard's theorem assuring that for smooth f , almost all values are critical values.

Theorem 3 (Sard theorem). *If f is a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^n$ with $n < m$, then almost all c are not critical values.*