**DIFFERENTIAL GEOMETRY**

MATH 136

# Unit 1: What is differential geometry?

## Introduction

**1.1. Differential geometry** studies **manifolds** using algebra and analysis. All manifolds can be embedded in some Euclidean space $\mathbb{R}^n$. Either as the kernel of a smooth map $f : R \subset \mathbb{R}^n \to \mathbb{R}^m$, or then as the image of a vector valued map $r : \mathbb{R}^k \to \mathbb{R}^n$. One-dimensional manifolds are **curves**, Two dimensional manifolds form **surfaces**. The kernel $\{f = 0\}$ of $f : \mathbb{R}^n \to \mathbb{R}$ is a hyper-surface of dimension $n - 1$. The image $f(R)$ of a function $r : R \subset \mathbb{R} \to \mathbb{R}^n$ is in general a curve. The kernel $\{f = 0\}$ of $f : \mathbb{R}^3 \to \mathbb{R}^2$ is often a **curve**.

**1.2.** Like in calculus, we are interested both in **global quantities** like **arc length** $\int_a^b |r'(t)| \; dt$ or **surface area** $\iint_R |r_u \times r_v| \; dudv$, where $\times$ is the **cross product** or **local quantities** like **curvature** $\kappa(t) = |r'(t) \times r''(t)|/|r'(t)|^3$ or **torsion** $\tau(t) = \det[r'(t), r''(t), r'''(t)]/|r' \times r''|^2$ for curves. We will define such concepts in arbitrary dimensions using the metric tensor.

**1.3.** Curvature plays an central role in differential geometry. We will define it using the metric tensor $g$ and verify that it is independent of the embedding in space. This is the **Theorema egregium**, the "great theorem" of Gauss from 1827. Riemannian geometry is based the idea of doing geometry on a manifold can be done also without having to embed it into an ambient emerged in an inaugural lecture of Riemann in 1854. The theory is used heavily in Einstein's 1915 theory of general relativity. Schwarzschild found a black hole solution in 1916. A 100 years later, gravitational waves from black hole mergers have been observed. The theory is well tested experimentally.

**1.4.** One goal of differential geometry is to investigate relations between **local quantities** and **global properties**. An important example to observe what happens if curvature is integrated up. We are especially interested in quantities that do not depend on the metric, like the **Euler characteristic**. Here are examples which will appear early in this course: for a planar closed curve in $\mathbb{R}^2$ one can define the **signed curvature** as $\kappa(t) = (r'(t) \times r''(t))/|r'(t)|^3$. We will then see the **Hopf Umlaufsatz** $\int_C \kappa(t) \; dr(t) = 2\pi$. For a two-dimensional surface with $g$ holes of Euler characteristic $\chi(G) = 2 - 2g$, one has the **Gauss-Bonnet theorem** $\iint_R K(x) \; dV(x) = 2\pi\chi(G)$. We also want to understand **geodesics**, curves that locally minimize length. One can start geodesics into any direction $v/|v|$ and let it run for a distance $|v|$. This produces the **exponential map** $\exp_p$, a map from the tangent space $T_pM$ of a point $p$ to the

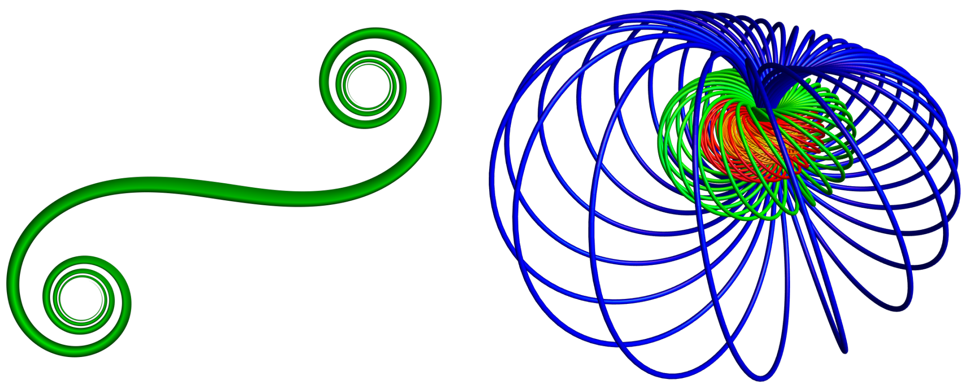

FIGURE 1. To the left, we see a curve called Euler curve, to the right, we see a visualization of the 3 dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. It is foliated by 2 dimensional flat tori. We can not embed flat tori in $\mathbb{R}^3$ but we can in $\mathbb{R}^4$.

manifold. If $S_r$ is the sphere of radius $r$ in $\mathbb{R}^2$, then the image $W_r(t) = \exp_p(S_r)$ is called the **wave front** at $p$. These waves can become complicated for large $r$. This can also be studied on polyhedra. We expect wave fronts to become dense in the manifold, except for very special cases like the round sphere.

**1.5.** Differential geometry extends curve and surface theory also to arbitrary dimensions. But also there, the low dimensional case plays a role. In relativity for example, one looks a paths in a space time manifold. Two dimensional sheets allow then to define **sectional curvature** and to use it to define a **curvature tensor** or **scalar curvature**. Some pretty cool analysis comes in when extremizing functions. Shortest curves between two points produce geodesics. Extremizing the total scalar curvature produces the Einstein equations. **General relativity** studies solutions of these equations. It tells how matter bends space and how matter moves in this space.

**1.6.** Differential geometry works well also in the discrete. It is part of calculus without limits. It is motivated by worries about the foundations of mathematics or because of technical difficulties. Some flavors of physics attempt to understand gravity for distances are beyond the Planck scale. The discrete world becomes a laboratory for experiments. In the discrete, many surprises await. The Gauss-Bonnet theorem - one of the mountain peak in this course - is much easier to handle in the discrete and works in arbitrary dimensions and on arbitrary networks. An other big surprise is that level surfaces in manifolds are always manifolds, if the level surface is not empty. Manifolds are in the discrete axiomatically defined using recursion with respect to dimension and does not use the continuum. The level set regularity fails in the continuum: the figure eight surface $x^2 - y^2 - z^2 - x^4 = 0$ is not a manifold. Also geodesics can be defined elegantly in discrete manifolds without invoking calculus of variations, or tensor calculus.

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