DIFFERENTIAL GEOMETRY

MATH 136

10th Homework

GR for one-dimensional space

Problem 1: A -manifold (M, g) of Lorentzian signature $(-,+)$ without boundary is a model for a space-time universe with one dimensional space.
a) Check that the Gauss-Bonnet proof for $\int_M K\, dV = 2\pi\chi(M)$ still works in the Lorentzian case. What is the general Gauss-Bonnet theorem for space time surfaces?
b) Why does a Lorentzian manifold of signature $(-,+)$ admit a global orthogonal frame bundle X? Conclude that a closed connected surface of Lorentzian signature $(-,+)$ needs to be a torus or Klein bottle.

Problem 2: a) Verify that on the flat torus with signature $(-,+)$ that is $g = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ at all points, the geodesics are lines $x(t) = \pm t + c \bmod 1$.
b) Look at the non-flat Lorentzian torus with $g = \begin{bmatrix} 0 & 1 \\ 1 & \sin(u) \end{bmatrix}$. Check that $K = -\sin(u)/2$.

Problem 3: a) Continuing with 2) check that null directions either satisfy $y' = 0$ or $y' = -2/\sin(x)$ and that the geodesic equations are $\ddot{x} + \cos(x)\dot{x}\dot{y} + \sin(x)\cos(x)\dot{y}^2/2 = 0, \ddot{y} - \cos(x)\dot{y}^2/2 = 0$. b) Conclude that $x(t) = 0, y(t) = -2\log(1-t)$ is a solution. As $t \to 1$ it winds around the torus infinitely often and has no continuous extension at $t = 1$.
c) Conclude that a smooth Lorentzian metric on a compact torus can have incomplete null geodesics. There is no Hopf-Rynov. Why did the proof from the Riemannian case not work here?

Problem 4: a) Check that in dimension 2, we have $R_{mikj} = K[g_{mk}g_{ij} - g_{mj}g_{ik}]$, where K is the Gaussian curvature. (You can use that there is a Lorentzian orthonormal basis so that $g_{00} = -1, g_{11} = 1, g_{01} = g_{10} = 0$).
b) Verify that a 2-dimensional Pseudo Riemannian manifold (M, g) is an Einstein manifold, meaning that the Einstein tensor $G = R - Sg/2$ vanishes.
c) Check that the example in problem 3) shows that the equation $R = \lambda g$ for a constant λ is in general false for an Einstein surface.

Fünf Variationen, See code on website

In each of these problems, just run the code and answer a short question with a number each to see whether your run has completed successfully.

Problem 5a): Verify that the Riemannian manifold (M, g) defined by the parametrization

$$r(u, v) = [(5 + 2\cos(v))\sin(u), (5 + 2\sin(v))\sin(u), \cos(v)]$$

satisfies the Einstein equations

$$R - \frac{1}{2}Sg = 0$$

where R is the Ricci tensor, where S is the scalar curvature and where g is the first fundamental form aka Riemannian metric. What is the scalar curvature S for $v = \pi/2$?

Problem 5b): Verify that the Schwarzschild metric $(M = \mathbb{R}^4, g)$ with

$$g = \begin{bmatrix} \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\phi) \end{bmatrix}$$

satisfies the Einstein equations

$$R - \frac{1}{2}Sg = 0$$

What is the Riemann tensor entry R_{1122} for $r = 1, M = 1$?

Problem 5c): The Reissner-Nordstroem metric

$$g = \begin{bmatrix} -\frac{e^2}{r^2} + \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\frac{e^2}{r^2} - \frac{2M}{r} + 1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix}$$

is a static solution of the Einstein-Maxwell equations

$$R - \frac{1}{2} S g = T \ .$$

But it is not a vacuum. It describes the field of a charged, non-rotating body of mass M and charge e. An example is a charged black hole. What is the entry T_{44} for $e = 1, r = 1, \phi = \pi/3$?

Problem 5d): Verify that the metric on $SU(2) = S^3$ given by the parametrization

$$r = [\cos(u)\cos(w), \sin(u)\cos(w), \cos(v)\sin(w), \sin(v)\sin(w)]$$

satisfies vacuum Einstein equations $R - 2g = 0$. What is S?

Problem 5e): Finally check that the pseudo sphere given by the parametrization

$$r = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v) + \log\left(\tan(\frac{v}{2})\right)]$$

satisfies the vacuum Einstein equations. What is the curvature K?

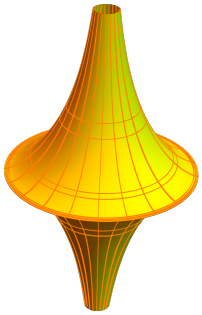


FIGURE 1. The Pseudo sphere

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