DIFFERENTIAL GEOMETRY

MATH 136

9th Homework

This is the 9th homework. It is due Friday, November 13th:

Problem 1: A) What kind of tensors are we dealing with?
a) The map on $\mathbb{R}^3 \times \mathbb{R}^3$ which assigns to (v, w) the number $v \cdot w$.
b) The cross product $(v, w) \in E = \mathbb{R}^2$ producing $v \times w \in \mathbb{R}$.
c) The map on $E = M(3, 3)$ which assigns to $A \times B \in E \times E$ the number $\mathrm{tr}(A^T B)$.
d) The determinant $(v_1 \dots, v_n) \in (\mathbb{R}^n)^n \to \det(A) \in \mathbb{R}$, where A has $v_1 \dots v_n$ as column vectors.
e) The determinant $(v_1 \dots, v_n) \in (\mathbb{R}^*)^n \to \det(A) \in \mathbb{R}$, where A has $v_1, \dots v_n$ as row vectors.
B) a) Let $E = \mathbb{R}^3$. Why is the map $f : E \times E \to \mathbb{R}$ defined by $(v, w) \to v \cdot v + w \cdot w$ not a tensor?
b) Why is the map $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $f(v, w)) = v \cdot w + 1$ not a tensor?

Problem 2: A) What type of tensor field do we have?
a) The calculus gradient field $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$.
b) The Jacobian of a function $df = [f_x, f_y, f_z]$
c) The second fundamental form II at a point.
d) The shape operator A at a point like $(Av)^k = \sum_j A^k_j v^j$.
e) The inverse I^{-1} of the first fundamental form. Written g^{ij}.
B) a) A (p, q) tensor field T not only just a smooth map that attaches a (p,q) tensor at every point. It also has to transform properly: we require $T(fX_1^*, X_2^*, \dots X_p^*, Y_1, \dots Y_q) = fT(X_1^*, X_2^*, \dots X_p^*, Y_1, \dots Y_q)$ for every smooth function f (and similarly for any of the p+q arguments). Derive that his means in coordinates

$$T^{a_1 \cdots a_p}_{b_1 \cdots b_q} = \frac{\partial y^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{a_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial y^{b_1}} \cdots \frac{\partial x^{j_q}}{\partial y^{b_q}} T^{i_1 \cdots i_p}_{j_1 \cdots j_q}$$

using p Jacobians and q inverse of Jacobian matrices.
Use this to see that the second derivative $r_{u^i u^j}$ is not a tensor field and also that the Christoffel symbols $\Gamma_{ijk} = r_{u^i u^j} r_{u^k}$ is not a tensor field.

Problem 3: a) Prove that $SU(2)$ is a manifold by explicitly giving the charts.
b) We have seen that we can describe a 2-sphere with 2 charts. Show that any compact connected 2-manifold can be described using 3 disc charts.

Problem 4: A discrete m manifold M was called a m-sphere, if it becomes contractible after removing one of its vertices. Inductively, a graph was called contractible, if we can find a vertex v such that both $G \setminus v$ and $S(v)$ are contractible. We defined the join of two graphs to be the disjoint union in which every vertex of the first is connected to every vertex of the second.
a) Prove that if M is a discrete p sphere and N is a discrete q sphere, then the join $M * N$ is a discrete $p + q + 1$ sphere.
b) Illustrate this in to see that the join of a 1 sphere with a 0 sphere is a 2 sphere.

Problem 5: The following code allows you to experiment with level sets in discrete manifolds. The host manifold is a discrete 4-manifold, the join of a 2-sphere and a 1-sphere. Running the code builds a random function from the vertex list to $\{1, 2, 3\}$. It defines a co-dimension 2 manifold.
a) Run the code as it is, report the numbers V,E,F as well as the Euler characteristic of M_f.
b) Change the code and see what happens if the function takes 4 values.
c) Build a 5 manifold as the join of two 2-manifolds and build a 3-manifold by taking a function taking 3 random values.
d) Report the curvature values of your 3-manifold.
e) Report the curvature vales of a 4 manifold by taking a function taking 2 values on the 5 manifold in c).

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Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]]; w[x_]:=-(-1)^k;
R[G_,k_]:=Module[{},R[x_]:=x->RandomChoice[Range[k]]; Map[R,Union[Flatten[G]]]];
F[G_]:=Delete[BinCounts[Map[Length,G]],1]; Euler[G_]:=F[G].Table[w[k],{k,Length[F[G]]}];
Surface[G_,g_]:=Select[G,SubsetQ[#/.g,Union[Flatten[G] /. g]] &];
S[s_,v_]:=VertexDelete[NeighborhoodGraph[s,v],v]; Sf[s_,v_]:=F[Whitney[S[s,v]]];
Curvature[s_,v_]:=Module[{f=Sf[s,v]},1+f.Table[(-1)^k/(k+1),{k,Length[f]}]];
Curvatures[s_]:=Module[{V=VertexList[s]},Table[Curvature[s,V[[k]]],{k,Length[V]}]];
J[G_,H_]:=Union[G,H+Max[G]+1,Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1}],0]]]];
ToGraph[G_]:=UndirectedGraph[n=Length[G];Graph[Range[n],
  Select[Flatten[Table[k->l,{k,n},{l,k+1,n}],1],(SubsetQ[G[[#[[2]]]],G[[#[[1]]]]])&]]];
Barycentric[s_]:=ToGraph[Whitney[s]];

G=J[Whitney[Barycentric[CompleteGraph[{2,2,2}]]],Whitney[CycleGraph[7]]]; (* J=Join *)
g=R[G,3]; H=Surface[G,g]; (* A codimension 2 manifold in the 4-sphere G=Oct * C_7 *)
Print["EulerChi=",Euler[H]]; Print["Fvector:",F[H]]; s=ToGraph[H]; GraphPlot3D[s]
Print["Gauss-Bonnet-Check:"]; Print[Total[Curvatures[s]]==Euler[H]];
Print["Curvature-Values:--"]; Print[Union[Curvatures[s]]];
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