

DIFFERENTIAL GEOMETRY

MATH 136

Homework 7

This is the Halloween homework. It is due Friday, October 30:

Problem 13.1: To warm up to Greens theorem, solve the following problems: a) Use Green to compute the area of the region $|x|^{2/3}/a^2 + |y|^{2/3}/b^2 \leq 1$.
b) Assume F is a smooth 1-form on a genus 3 surfaces M , what can you say about $\iint_M dF$?
c) What can you say about $\iint_M K \, dA$ if M is a 2-manifold with parametrization r ?
d) We know that $\iint_M K \, dA = 4\pi$ for a sphere! Conclusion?

Problem 13.2: a) Green's theorem tells that if $R \subset \mathbb{R}^2$ is a region and $X = [P, Q]$ is a vector field in the plane, then $\iint_R \text{curl} X \, dudv = \int_{\delta R} X(r(t)) \cdot r'(t) \, dt$ where δR is the boundary. Look up and write down a proof of this.
b) Look up the discrete Green theorem and give a proof.

1

Problem 13.3: Verify here that Stokes theorem on $S = r(R)$ can be reduced to Green on R :

$$\iint_R \text{curl}(F) r_u \times r_v \, dudv = \iint_R \text{curl}(X) \, dudv$$

Assume $F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is a vector field in space. Prove the important formula

$$\text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u .$$

As we have seen in class, this implies that the 2D field $X = [F \cdot r_u, F \cdot r_v]$ satisfies $\text{curl}(X) = F_u \cdot r_v - F_v \cdot r_u$.

¹<https://people.math.harvard.edu/~knill/teaching/math22b2022/handouts/lecture33.pdf>

Problem 13.4: We see on Wednesday the completion that X is intrinsic. Verify that also the second component of $X = [z \cdot w_u, z \cdot w_v]$ can be expressed from I alone.

Problem 13.5: Below you see Gauss's original statement of the theorem Egregium translated into English. Explain what he means with "developing a surface upon any other surface" and why it is not possible for example to find a map of the earth in $\mathbb{R}^3$ which preserves distances.

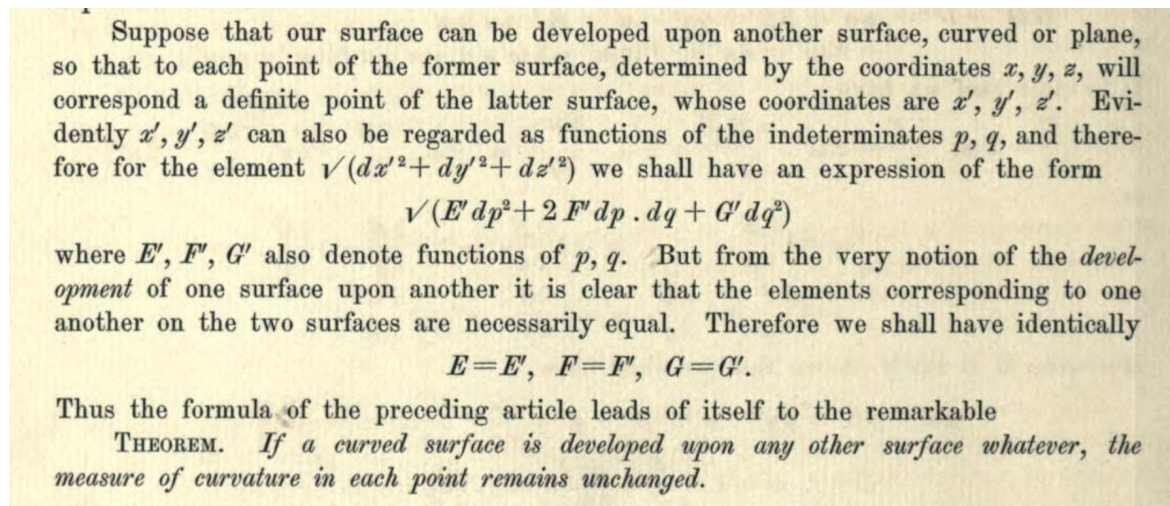


FIGURE 1. Gauss original statement translated into English. Source: Wikipedia