

DIFFERENTIAL GEOMETRY

MATH 136

Homework 6

This is the sixth homework. It is due Friday, October 23, 2026.

Problem 1: a) Compute the curvatures of the Pentakis Icosidodecahedron (Golden Fullerene) and verify Gauss-Bonnet in this case.
b) Divide each face of a dodecahedron into 5 triangles, then compute the curvatures and verify Gauss Bonnet.

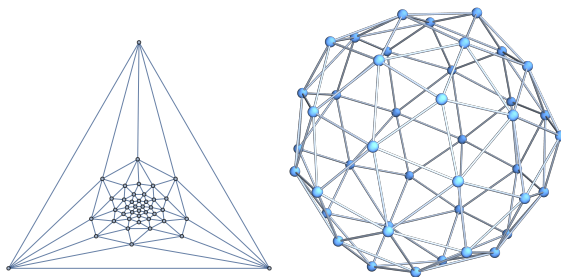


FIGURE 1. The Pentakisicosahedron is a 2-sphere with 42 vertices, 120 edges and 80 triangles. In chemistry it appears as $Au_{30}Si_{12}$.

Problem 2: a) The effort to hike along a path is measured by $F(x, \dot{x}) = x^2 \dot{x}^2$ rather than the kinetic energy $\dot{x}^2$. Hikes are easy initially but get harder later on. What is the best strategy to reach from $x(0) = 0$ to $x(1) = 1$? Go slow first or go fast first? In order to find the best way, solve the Euler-Lagrange problem to minimize the action

$$E(x) = \int_0^1 F(x, \dot{x}) dt$$

for $F(x, \dot{x}) = x^2 \dot{x}^2$ with $x(0) = 0, x(1) = 1$.

b) Do the same for $F(x, \dot{x}) = x \dot{x}^2$.

Problem 3: Look at the paraboloid $r(u, v) = [u, v, u^2 + v^2]$.

a) Compute all the Christoffel symbols Γ_{ijk} . These are 8 terms. (Do this by hand).

b) Now compute all the Christoffel symbols Γ_{ij}^k . (Also here, do this by hand. There is Mathematica code which can be used for the last problem).

Problem 4: We draw some wave fronts $W_r(p)$ on the flat Clifford torus $\mathbb{R}^2/\mathbb{Z}^2$ (Pac-Man square). This manifold can be realized as all point (x, y) in $\mathbb{R}^2$, where points $(x, y), (x + n, x + m)$ identified if $n, m \in \mathbb{Z}$. Work with the point $p = (1/2, 1/2)$.

a) Draw the wave front $W_{1/2}(p)$.

b) Draw the wave front $W_1(p)$.

c) Draw the wave front $W_5(p)$.

You are welcome to become physical part c and use scissor, ruler and compass to do that).

Problem 5: The torus

$$r(u, v) = ((a + b \cos(v)) \cos(u), (a + b \cos(v)) \sin(u), b \sin(v)) ,$$

has the metric

$$g_{11} = (a + b \cos(v))^2$$

$$g_{22} = b^2$$

$$g_{12} = g_{21} = 0$$

Use the following example code (done for the sphere) to compute all the Christoffel symbols Γ_{ij}^k . Make sure to simplify.

```
r={Sin[v] Cos[u], Sin[v] Sin[u], Cos[v]};
ru=D[r, u]; rv=D[r, v];
n=Cross[ru, rv]; n=n/Sqrt[n.n];
nu=D[n, u]; nv=D[n, v];
drT={ru, rv}; dr=Transpose[drT];
g=drT.dr; gi=Inverse[g];
dnT={nu, nv}; dn=Transpose[dnT];
h=-dnT.dr; e=dnT.dn;
K=Det[h]/Det[g];
X={u, v}; d=2;
c[i_, j_, k_] := (D[g[[j, k]], X[[i]]]
+ D[g[[k, i]], X[[j]]]
- D[g[[i, j]], X[[k]]]) / 2;
Christoffel[i_, j_, k_] := Sum[gi[[k, l]] * c[i, j, l], {l, d}];
S=Table[Simplify[Christoffel[i, j, k]], {i, d}, {j, d}, {k, d}];
TableForm[S]
```