

DIFFERENTIAL GEOMETRY

MATH 136

Homework 5

Problem 1: Prove the matrix identity $III - 2HII + KI = 0$, where $H = (\lambda + \mu)/2$ and $K = \lambda\mu$ and λ, μ are the eigenvalues of A .

Problem 2: The following Mathematica sample code computes the first and second fundamental form and curvature for a torus. Modify the code to get integrate up the curvature in the case $a = 1$ or $a = 2$. The code below is a template fore $a = 0$. You will see different results. Explain.

$$r(u, v) = \begin{bmatrix} (3 + \sin(v) + a \cos(6u)) \cos(u) \\ (3 + \sin(v) + a \cos(6u)) \sin(u) \\ \cos(v) \end{bmatrix}.$$

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r={ (3+Sin[v]) Cos[u], (3+Sin[v]) Sin[u], Cos[v] };
ru=D[r,u]; rv=D[r,v];
n=Cross[ru,rv]; n=n/Sqrt[n.n];
nu=D[n,u]; nv=D[n,v];
drt={ru,rv}; dr=Transpose[drt];
dnt={nu,nv}; dn=Transpose[dnt];
g=drt.dr; h=-dnt.dr; e=dnt.dn;
A=Inverse[g].h; H=Tr[A]/2; K=Det[A];
dV=Sqrt[Det[g]]; f=Simplify[K dV];
NIntegrate[f, {u,0,2 Pi},{v,0,2 Pi}]
```



FIGURE 1. A 6-star torus for $a = 1$. You also look at the case $a = 2$. The example code covers the case $a = 0$.

Problem 3: a) Compute the curvatures of the Pentakis Icosahedron (Golden Fullerene) and verify Gauss-Bonnet in this case. b) Divide each face of a cube into 2 triangles, then compute the curvatures and verify Gauss Bonnet.

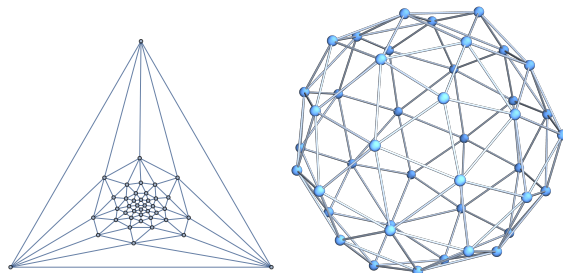


FIGURE 2. The Pentakisicosahedron is a 2-sphere with 42 vertices, 120 edges and 80 triangles. In chemistry it appears as $Au_{30}Si_{12}$.

Problem 4: A 3-manifold is a finite simple graph for which every unit sphere is a 2-sphere. The classification of 3-manifolds is much more difficult than the classification of 2-manifolds.

- Look up the 600 cell and the 16 cell and show that they are 3-manifolds.
- The curvature of a 3-manifold is defined as $K(v) = 1 - d_0(v)/2 + d_1(v)/3 - d_2(v)/4$, where $d_0(v)$ is the number of vertices of $S(v)$ and $d_1(v)$ is the number of edges of $S(v)$ and $d_2(v)$ is the number of triangles in $S(v)$.
- Compute the curvature of the 600 cell and the 16 cell.

Problem 5: A **2-manifold with boundary** is a graph such that every unit sphere is either a circular graph C_n with $n \geq 4$ vertices or a path graph P_n with $n \geq 2$ vertices. The former points are called interior points. The curvature at a general point is $K(v) = 1 - |S_0(v)|/2 + |S_1(v)|/3$, where $S_0(v)$ is the set of vertices of $S(v)$ and $S_1(v)$ is the set of edges in $S(v)$. a) Verify from this definition that for a manifold with boundary, the curvature of a boundary point is $1/2 - |S_1(v)|/6$ and the curvature is $K(v) = 1 - |S_1(v)|/6$ for interior points. b) Check that the graph complement G of C_7 is a 2-manifold without interior. It implements the smallest Möbius strip. Verify that all curvatures of G are zero.