

DIFFERENTIAL GEOMETRY

MATH 136

Homework 4

This is the fourth homework. It is due Friday, October 2nd.

Problem 1: There is a **discrete Hopf Umlaufsatz** for polygons.

a) Assume first we have a simple convex polygon with n vertices. Define the curvature at the vertex v_k to be κ_k which is the outer angle $\pi - \alpha_k$ where α_k is the angle you have defined in third grade for polygons. The discrete Hopf Umlaufsatz tells $\sum_{k=1}^n \kappa_k = 2\pi$. Prove this.

b) Now formulate the general (not necessarily convex) case. Define suitable curvatures (which are no more positive now in general) such that the result works.

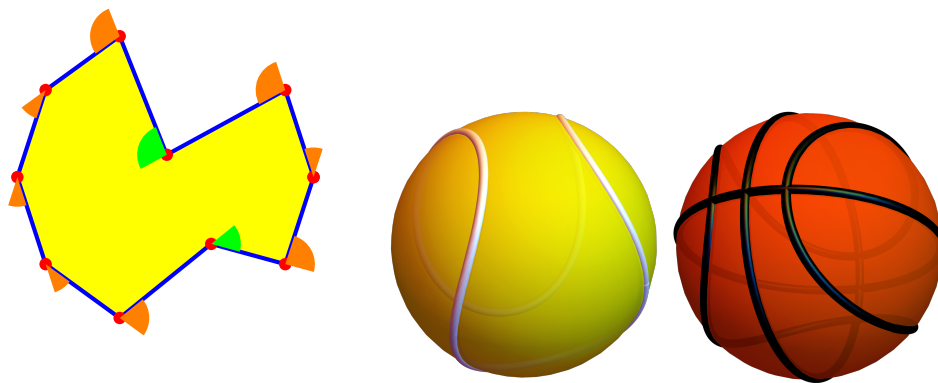


FIGURE 1. In the Umlaufsatz for polygons, curvatures can get positive or negative. The tennis ball and basket ball curve.

Problem 2: Check the four vertex theorem in the example $r(t) = 4[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)]$. The expressions for $\kappa(t)$ and $\kappa'(t)$ are not that bad. Plot the function $\kappa(t)$ and find the critical points, the roots of κ' .

Problem 3: For parameters a , define $c = 2\sqrt{a}$ and the curve $r(t) = [a \cos(t) + \cos(3t), a \sin(t) - \sin(3t), c \sin(2t)]$. For $a = 2$ one has the **tennis ball curve**, for $a = 1/2$ the **base ball curve** and for $a = 1.8$, the **basket ball curve** (Basketballs have two additional grand circles).

- Verify that these curves are located on a sphere.
- Look up the tennis-ball theorem, state its content, then write down the main idea on how the theorem is proven.

Problem 4: a) Compute the first and second fundamental form for the surface

$$r(u, v) = \begin{bmatrix} u \\ u^2 - v^2 \\ v \end{bmatrix}.$$

In this problem, we do not want you to use a computer algebra system. You need to write down especially all the matrices dr, dn and A .

- Use a) to compute the curvature and especially the curvature at $(0, 0, 0)$.

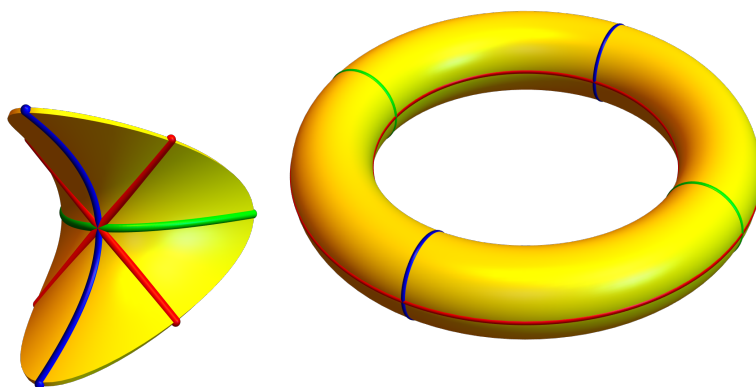


FIGURE 2. A surface with negative curvature. The torus for problem 4-5.

Problem 5: Compute the first and second and third fundamental form for the torus $r : R \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$r(u, v) = [(5 + \cos(v)) \cos(u), (5 + \cos(v)) \sin(u), \sin(v)],$$

where $u, v \in [0, 2\pi)$. We want you to write down explicit expressions for the matrices dr, dn . This problem again can be solved by hand, but you are allowed here to use a computer algebra system to assist you.