

DIFFERENTIAL GEOMETRY

MATH 136

Homework 3

This is due Friday, September 25.

Problem 1: a) Look at the Hopf Umlaufsatz in the example of an asteroid

$$r(t) = [\cos^3(t), \sin^3(t)] .$$

First verify that $|r'(t)| = |3 \cos(t) \sin(t)|$ and $\kappa(t) = -(2/3)/|\sin(2t)|$ then compute the rotation index $\int_0^{2\pi} \kappa(t) |r'(t)| dt / (2\pi)$. While your result complies with the Hopf Umlaufsatz, there is something strange going on given how you rotate counterclockwise around the region. Figure it out!

b) For which c is the curve $r(t) = [\cos(t), \sin(t) + c \sin(3t), 0]$ a simple regular curve?

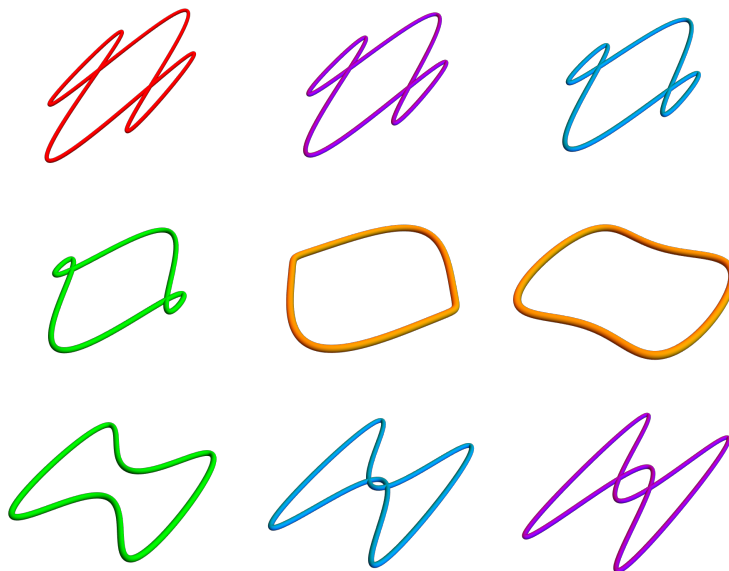


FIGURE 1. A few cycloid curves described in problem 1. There are cases for which the curve is simple and curves for which it is not.

Problem 2: a) Determine from each of the spaces whether they are linear spaces. $Gl(n, \mathbb{R})$ is the space of all invertible $n \times n$ matrices.

Space	linear	not linear
$SO(n)$		
$so(n)$		
$SU(n)$		
$su(n)$		

b) Check that if $x(t)$ is a differentiable curve in $SO(n)$, then $x(t)$ satisfies the differential equation $x'(t) = A(t)x(t)$, where $A(t) \in so(n)$, the space of skew-symmetric matrices.

c) Show that $A(t) = A$ is a constant skew symmetric matrix, then the **matrix exponential** $Q(t) = e^{At}$ is an orthogonal matrix. What is this matrix $Q(t)$ in the case $n = 2$?

Problem 3: a) Verify that the helix

$$r(t) = [\cos(at), \sin(at), bt]$$

has constant curvature and torsion. What are the values?

b) Now prove that if a curve has constant curvature and torsion, it must be a helix.

Problem 4: Verify that the textbook formulas

$$\kappa = |r' \times r''|/|r'|^3, \quad \tau = (r' \times r'') \cdot r''' / |r' \times r''|^2.$$

have the property that if $r(t)$ is parametrized by arc length, then κ simplifies to $\kappa = T' \cdot N = e'_1 \cdot e_2$ and τ simplifies to $\tau = N' \cdot B = e'_2 \cdot e_3$.

Problem 5: a) Look up and write down a proof that if $F(t, x)$ is a differentiable function from $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$, then there exists an open interval $(-a, a)$ and a unique path $x(t) : (-a, a) \rightarrow \mathbb{R}^n$ such that $x' = F(t, x)$ and $x(0) = x_0$. We want you to write down the proof in the differentiable case which is a bit more special than the usual assumption assuming a Lipschitz property for F .

b) Justify that if $x(t)$ stays bounded meaning that there is constant such that $|x(t)| \leq M$ for all t , then the solution exists for all t . We call this a global solution. c) Now conclude that if $Q' = K(t)Q$ is a differential equation for a matrix $Q(t)$ with skew symmetric $K(t)$, then there is a global solution $Q(t)$ in $SO(n)$