DIFFERENTIAL GEOMETRY

MATH 136

1. Homework, Due Fri September 11

Problem 1.1: A "Samurai bartender" cuts ice into a **super-egg**. Super eggs are super ellipsoids with an axis of symmetry. The super egg of our Samurai is

$$f(x,y,z) = (\sqrt{x^2+y^2}/6)^8 + (z/5)^8 = 1 \ .$$

a) Verify that this super egg is a manifold. No computer algebra in a).
b) Parametrize the surface as $r(\phi,\theta)$ and find its surface area A.
c) Find the volume V of the interior numerically.
d) Check that the isoperimetric quotient $Q = 36\pi V^2/A^3$ is between the value you get for the sphere and the value you get for the cube.

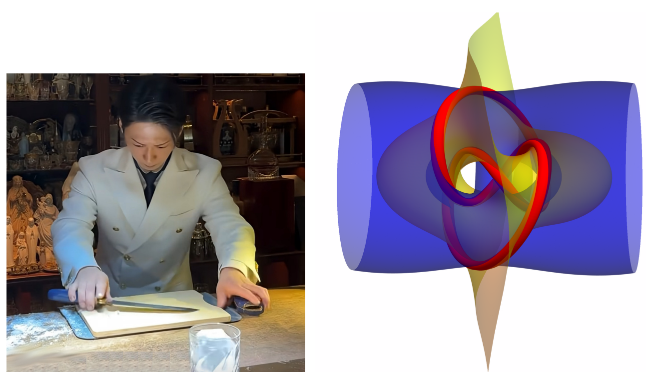

FIGURE 1. Samurai bartender Yuzo Kumai cutting a super egg for problem 1). To the right we see the three foil knot obtained by intersecting two polynomial surfaces given in problem 2)

Problem 1.2: A **knot** is the image C of an injective C^1 map $r(t) = [x(t), y(t), z(t)]$ from the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ to space $\mathbb{R}^3$.
The concrete $r(t) = [x(t), y(t), z(t)] = [4\cos(3t), 4\sin(3t), 3\cos(2t)]/(5 - 3\sin(2t))$ defines a knot.
a) With $R = x^2+y^2+z^2$, check that C is the intersection of two polynomial surfaces $p(x,y,z) = 0, q(x,y,z) = 0$ given by $p = 27(x^2-y^2)(R+1) - 160z^3 + 120z(R-1)^2$ and $q = 27xy(R+1) - 120z^2(R-1) + 10(R-1)^3$.
b) Prove that $p = 0$ and $q = 0$ both are 2-manifolds and that $f(x,y,z) = [p(x,y,z), q(x,y,z)] = [0,0]$ is a 1-manifold.

Problem 1.3: The fact that knots in 4D are trivial has been popularized by Leonard, Sheldon and Penny in one of the Big Bang Theory episodes of season 11. Look at the knot C defined by $r(t) = [x(t), y(t), z(t), w(t)] = [4\cos(3t), 4\sin(3t), 3\cos(2t), 3\sin(2t)]$ in four dimensional space.
a) Verify that the knot lives simultaneously on the **3-sphere** $S = x^2 + y^2 + z^2 + w^2 = 25$ as well as on the **2-torus** $\{x^2 + y^2 = 16\} \cap \{z^2 + w^2 = 9\}$.
b) Explicitly deform C to the trivial knot.

FIGURE 2. Sheldon and Penny discussing strings and sheets. Levent Alpoge to the right.

Problem 1.4: This summer, a former Harvard math concentrator Levent Alpge shocked the world by triggering the top model of Antropic "fable" to find a counter example to the real Jacobian conjecture: a polynomial map $f(x,y,z) = [f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)]$ in $\mathbb{R}^3$ with constant nonzero determinant is globally injective. The example is

$$f(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} (1+xy)^3 z + y^2(1+xy)(4+3xy) \\ y + 3x(1+xy)^2 z + 3xy^2(4+3xy) \\ 2x - 3x^2 y - x^3 z \end{bmatrix} .$$

Check that $f(A) = f(B)$ for $A = (0, 0, -1/4)$ and $B = (1, -3/2, 13/2)$ and that df has constant determinant.

Problem 1.5: Our homework policy requires you to to work on problems on your own without giving it to the AI. In this problem we allow auditing AI-generated mathematics. Go chat with your favorite AI companion and try to make them find a counter example of the Jacobian conjecture for polynomial maps $f(x,y) = [p(x,y), q(x,y)]$ with two polynomials. Log the interesting parts of this conversation in a paragraph or two. Do not just parrot the chat (or let the bot summarize it for you) but report the outcome in your words.