DIFFERENTIAL GEOMETRY

MATH 136

Final Exam Paper

Assignment: due December 12, 2026

Write an expository paper about a topic in differential geometry of your choice.

Rules

- The paper must be written by yourself. No external human nor AI assistance is allowed in writing. Literature search is fine
- Acknowledge any references used: books, papers, web, discussions, the use of computer algebra.
- Aim for 4 or more pages with references. There can be illustrations. Also for illustrations, either do them yourself or give credit.

Grading criteria

(1) Adaptation to the course, notation
(2) Mathematical correctness
(3) Clarity, readability and elegance
(4) References and sources
(5) Originality or depth or surprise

The paper is 50 points, the in class exam is 50 points. In total: 100 points.

Topic suggestions

Pick or modify one of the following topics. If you prefer an other topic, check with Oliver. The topic needs to be related to this course - of course! This semester, I especially will weight more adaptation to the course, the ability to use notation and what we have learned here.

(1) **"Minimal surfaces"**
(2) **"Stability of curves defined by curvature and torsion."**
(3) **"Positive Curvature Manifolds"**. Examples of even dimensional manifolds for which all sectional curvatures are positive.
(4) **"Clifton-Pohl torus"** A pseudo Riemannian manifold on which the geodesic flow is not complete.
(5) **"Turning the sphere inside out"** A construction first envisioned by Smale.
(6) **"The Hopf manifold $S^2 \times S^2$."**
(7) **"Caustics in Riemannian geometry"**

(8) **"What is a Jacobi field?"**
(9) **"Differential geometry of evolutes."**
(10) **"Principal curvatures and umbillic points"**
(11) **"The symmetries of the Riemann tensor"** Like Bianchi identities.
(12) **"Chaotic geodesic flows"** (like on surfaces of constant negative curvature).
(13) **"My favorite Riemanninan manifold"**. i.e. projective plane, the Klein bottle, the hyperbolic plane, the projective 3-space $SO(3)$ (a 3 manifold) or $SU(3)$ (a 8 dimensional manifold).
(14) **"The Differentiable Jordan Curve theorem"**. Easier than the topological Jordan Curve theorem. Prove it. See Do carmo page 400.
(15) **"The Hopf-Rynov Theorem"**: Two points on a complete surface can be joined by a minimal geodesic. See Do Carmo page 338.
(16) **"The formulas of Codazzi-Meinardi"**. See section 4.C in Kuehnel.
(17) **"The world of discrete manifolds"**. Explore small examples of discrete manifolds, I.e. Klein bottles, projective planes, or higher genus Klein bottles.
(18) **"The Brachistochrone problem"** An example of a variational problem.
(19) **"The Hilbert Action"** Outline the proof that critical points of the Hilbert action are Einstein manifolds.
(20) **"What are Minimal Surfaces?"**. See chapter 3D in Kuehnel.
(21) **"Complex manifolds"** or **Kähler manifolds**.
(22) **"The complex projective plane $\mathbb{CP}^2$."**
(23) **"What is a Kalabi-Yau manifold?**
(24) **"How can Morse theory help to understand Riemannian geometry?"**
(25) **"What are Ruled surfaces?"** Section 3C in Kuehnel.
(26) **"The rigidity of the sphere"** a compact connected regular surface of constant curvature must be a sphere. There is a section in Do Carmo.
(27) **"Clairaut's relation for geodesics"** A conserved quantity for geodesics in surfaces of revolution".
(28) **"The isoperimetric inequality"**. There is an exposition in Do Carmo.
(29) **"Geodesic rigidity of Gauss"**: if p, q are two points on a manifold and if $\gamma_s(t)$ is a smooth family of geodesics connecting p, q, then all these geodesics must have the same length.
(30) **Gauss Lemma in Riemannian geometry**: why are geodesics are perpendicular to wave fronts.
(31) **"Exotic spheres?"** What are spheres with non-standard differential structure.
(32) **"What is Willmore energy?"** The boy surface is known as "Oberwohlfach surface". A theorem of Bryant-Kusner tells that the Boy surface minimizes the Willmore energy.
(33) **"Geometry of hyperbolic space"**. i.e. geodesics, in a 2 dimensional manifold of constant curvature -1. Gauss-Bonnet for polygons.
(34) **"Fenchel's theorem"** or the Fary-Milnor theorem about total curvature of curves. There is a section in Kuehnel.
(35) **"The fundamental theorem of Riemannian geometry"** a connection satisfying three axioms must be the Christoffel connection.
(36) **"Classifying 2-manifolds"**. Describe the classification of 2-manifold as a connected sum of tori or projective planes.
(37) **"Spherical space forms"**. See for example Theorem 7.30 in Kuehnel.

(38) **"What are lense spaces"**? What are spherical 3-manifolds?
(39) **"Synge's theorem"**: an even dimensional orientable positive curvature manifold is simply connected.
(40) **"The Poinare conjecture"**. How was the Poincare conjecture proven. Especially define the Ricci flow using notation we have used.
(41) **"Surgery of 3-manifolds"**. Explain how one can use knots to build 3-manifolds. Especially Dehn surgery and the Lickorish-Wallace theorem.
(42) **"The Kerr Metric"**. A solution of the Einstein equations. The Kerr metric" models a rotating non-charged black hole. The Kerr-Newman metric a rotating charged black hole.
(43) **"Bertrand-Puiseux formulas"**. Prove $|S_r(p)| = 2\pi r - \pi K\epsilon^3/3 + ...$
(44) **"Chaotic Billiards"**. The geodesic flow in manifolds with boundary. is called a billiard. This involves curvature.
(45) **"Riemannian metrics on Lie groups"**. Like on $SO(3)$ or $SU(2)$. How does one get a Riemannian metric on such spaces?
(46) **"The Willmore conjecture"**. What is the Willmore energy? Explain the conjecture and how it was solved.
(47) **"The tensor algebra"**. Especially the exterior algebra of skew symmetric tensors.
(48) **"Counting Periodic geodesics."**
(49) **"The isometric embedding problem R^3."**
(50) **Open questions in Riemannian geometry**
(51) **Look at the inverse Frenet problem in the case of piecewise linear periodic curvatures**.
(52) **"Curvature of convex polyhedra in four dimensions."**
(53) **"Curvature in a conformal metric.**

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