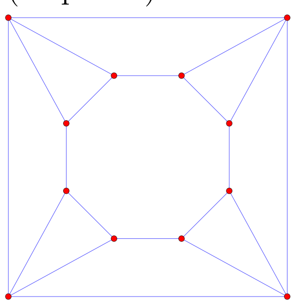
DIFFERENTIAL GEOMETRY

MATH 136

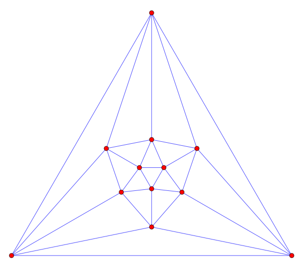
Final Part I (December 4, 2025)

Sign your name. You have 60 minutes. Closed book. No justifications are needed.

1) (10 points) Fill out the following table about graphs:

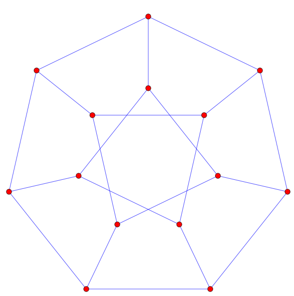


1

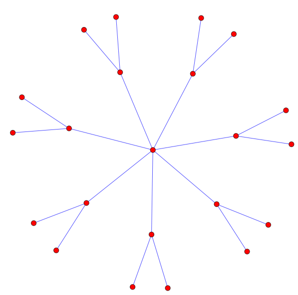


2

Graph	Contractible?	Manifold?
1		
2		



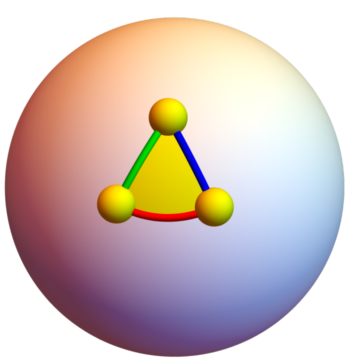
3



4

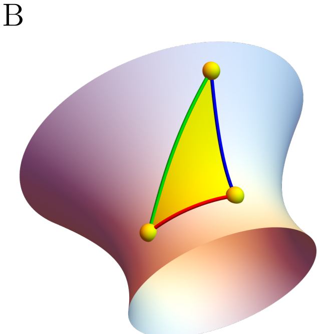
Graph	Contractible?	Manifold?
3		
4		

2) (10 points) You see triangles in a sphere of curvature 1 or part of a hyperbolic plane with curvature -1.The edges are geodesics and you are given the angles α, β, γ in radians. Find the area:



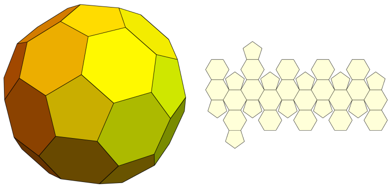
A

B

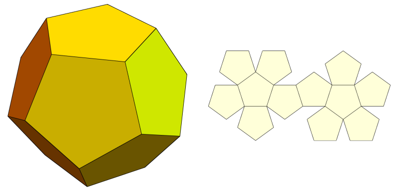


Triangle	α	β	γ	Area
A (in sphere)	$\pi/3$	$7\pi/18$	$7\pi/18$	
B (in pseudo sphere)	$\pi/6$	$\pi/2$	$5\pi/18$	

3) (10 points) You see polyhedra and their nets (which could be folded to build the polyhedron). In each case, check Descartes theorem, which is a Gauss-Bonnet theorem, where curvatures are located on vertices:



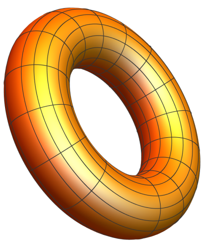
A



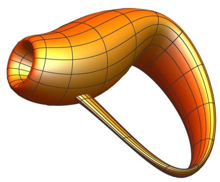
B

Polyhedron	excess $\kappa(v)$	$\|V\|$	$\|E\|$	$\|F\|$	$\chi(M)$	$\sum_{v \in V} \kappa(v)$
A						
B						

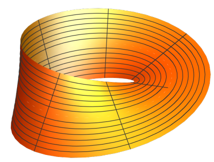
4) (10 points) Fill out the following table



1



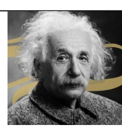
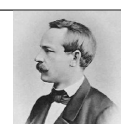
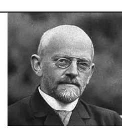
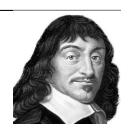
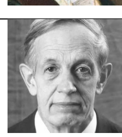
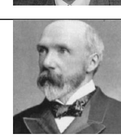
2



3

Surface	Orientable?	Manifold with boundary?	Manifold without boundary?	Euler characteristic
1				
2				
3				

5) (10 points) Name the mathematicians or fill in their achievements

Name	Picture	Known for	Name	Picture	Known for
		Relativity			Symbols Γ_{ijk}
Hilbert					Polyhedral Gauss-Bonnet
		Discrete relativity			$\frac{d}{dt} F_{\dot{x}} = F_x$
		Theorema Egregium	Hopf		
Nash					Fund. theorem of curves

6) (10 points) Name the objects

	$\Gamma_{ijk} = r_{u^i u^j} \cdot r_{u^k}$
	$A = I^{-1} II$
	$K = \det(A)$
Mean curvature	$H =$ ....
	$\kappa = T' \cdot N$
	$\tau = N' \cdot B$
	$\kappa_g(t) = (n \times \dot{x}) \cdot \ddot{x}$
Normal curvature	$\kappa_n =$ ....
Euler characteristic	$\chi(G) =$ ....
	$R^s_{ikj} = \frac{\partial}{\partial u^k} \Gamma^s_{ij} - \frac{\partial}{\partial u^j} \Gamma^s_{ik} + \sum_r \Gamma^r_{ij} \Gamma^s_{rk} - \sum_r \Gamma^r_{ik} \Gamma^s_{rj}$

7) (10 points) Match the following important identities or write in the formulas. If something needs to be filled to the left, use from the following: Cauchy-Binet identity, Gauss Bonnet for surfaces, Frenet equations, Stokes theorem, Greens theorem, Einstein equations, Identity for Fundamental forms, Hopf Umlaufsatz, Gauss-Bonnet for geodesic triangle, Karate Kick identity, Geodesic equations, Stokes theorem, Curvature identity.

	$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$
Einstein equations	
	$III - 2H II + K I = 0$
	$\int_C \kappa(t)\, dr(t) = 2\pi$
	$\int\int_U K dV + \sum_j \kappa_j = 2\pi$
Gauss-Bonnet for surfaces	
	$\ddot{x}^k + \sum_{i,j=1}^m \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0$
Curvature relations $\kappa_g, \kappa_n, \kappa$	
	$\lvert r_u \times r_v \rvert^2 = \lvert r_u \rvert^2 \lvert r_v \rvert^2 - \lvert r_u \cdot r_v \rvert^2 = \det(g)$
Karate-Kick identity	

8) (30 points)

1) The third fundamental form is a [ ] tensor.

2) The Schwarzschild metric describes the gravity of a [ ] hole.

3) A 90-90-X geodesic triangle on the unit sphere $x^2 + y^2 + z^2 = 1$ has area $\pi/2$. What is the angle X? [ ]

4) The Christoffel symbol Γ_{ij}^k is a tensor. True or false?

5) If F is a vector field which has $\text{curl}(F) = 0$, and G is the unit disk $x^2 + y^2 \leq 1, z = 0$ with boundary C then the line integral is $\int_C F\,dr =$ [ ] .

6) The shape operator matrix A is is a [ ] tensor.

7) A discrete manifold has the property that every unit sphere is a [ ] .

8) The 0-dimensional sphere has [ ] vertices and [ ] edges.

9) A connected 2-manifold with Euler characteristic 1 must be a [ ] .

10) The principle that action and length functional have the same critical points is called the [ ] principle.

11) It is possible that the Klein bottle has constant zero curvature. True or false?

12) The Möbius strip has Euler characteristic [ ] .

13) The projective plane is orientable. True or False? [ ] .

14) True or false: there are only finitely many closed periodic geodesics for a torus [ ] .

15) The Euler characteristic of a 2-manifold in terms of V, E, F is given by the formula [ ] .

16) If G is a discrete m manifold and $f : V(G) \to \{0, \ldots, k\}$ is a map, then M_f is a [ ] manifold or [ ] .

17) True or false: the Ricci tensor is symmetric: [ ] .

18) A key in the proof of the local Gauss-Bonnet theorem is [ ] 's theorem.

19) If M is $x^2/9 + y^2/4 + z^2/16 = 1$, then $\iint_M K \; dV =$ [ ] .

20) A 3-dimensional manifold has Euler characteristic [ ] .

21) The Euler's gem formula is [ ] .

22) The geodesic curvature of a geodesic is [ ] .

23) The normal curvature of a grand circle in a sphere of radius 5 is [ ] .

24) The geodesic curvature of a grand circle in a sphere of radius 5 is [ ] .

25) In a triangle on a sphere the sum of the angles is [ ] than π.

26) In a triangle on a flat torus, the sum of the angles is equal to [ ] .

27) Lambert's theorem deals with the sum of the angles of a triangle on a [ ] .

28) The sum of the angular defects $K(p)$ of a convex polyhedron is equal to [ ] .

29) True or False? Geodesics extremize the energy functional [ ] .

30) The Einstein equations can be derived by extremizing the [ ] functional.

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

Score: