DIFFERENTIAL GEOMETRY

MATH 136

Midterm part II. Due Friday 10/17/2025, midnight

Handwritten. No internet, no correspondence, no computer algebra system, closed book, no internet no AI. One single page of handwritten notes is allowed. Put your name on each page of your paper and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."** This part has 70 points. Part I done in class could give 30 points. In total, the full midterm counts 100 points.

Problem A (20 points): The graph $G = (V, E)$ in the picture is an example of a **positive curvature 2-manifold**.
a) (4 points) Verify that this is a 2-manifold. Describe how to do this.
b) (4 points) What does positive curvature mean for 2-manifolds?
c) (4 points) Compute all curvatures of G.
d) (4 points) Compute the sum of all curvatures of G.
e) (4 points) Count V, E and F and the Euler characteristic $\chi(G)$ of G.

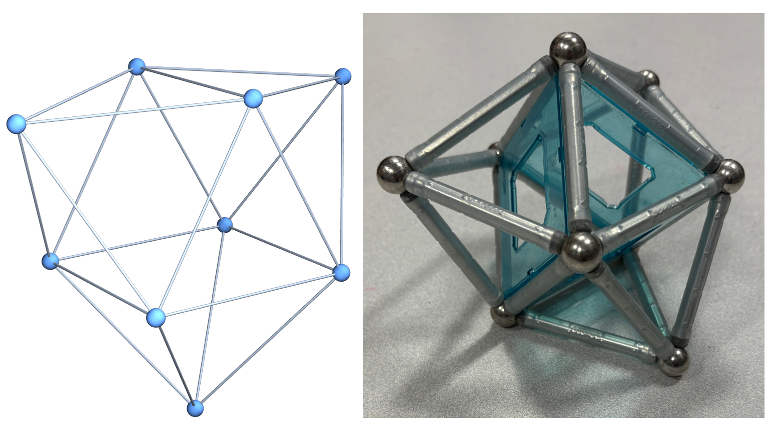

FIGURE 1. There are exactly 6 positive curvature manifolds of dimension 2, the octahedron and icosahedron are two examples. The species given here can be obtained by gluing three pyramids to a triangular cylinder. The picture to the right is a build-up with Oliver's geomag sticks.

Problem B (20 points): The **Cissoid of Diocles** C is the contour curve

$$f(x,y) = x^3 + xy^2 - y^2 = 0$$

It is an example of a **cubic algebraic curve**.

a) (2 points) Verify that C has the parametrization

$$r(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{t^2}{1+t^2} \\ \frac{t^3}{1+t^2} \end{bmatrix} .$$

b) (2 points) Is C a regular curve with the parametrization in a)?
c) (2 points) Is C a Frenet curve with the parametrization in a)?
d) (2 points) It C a closed curve ?
e) (2 points) Is C a simple curve?
f) (2 points) Is the curvature $\kappa(t)$ of C defined everywhere?
g) (2 points) Compute $df(x,y)$, then $df(r(t))$.
h) (2 points) What is $\frac{d}{dt} f(r(t))$?
i) (2 points) Use the chain rule to get $df(r(t))r'(t)$.
j) (2 points) Write down the arc length integral for $t \in [0,1]$. No evaluation is needed!

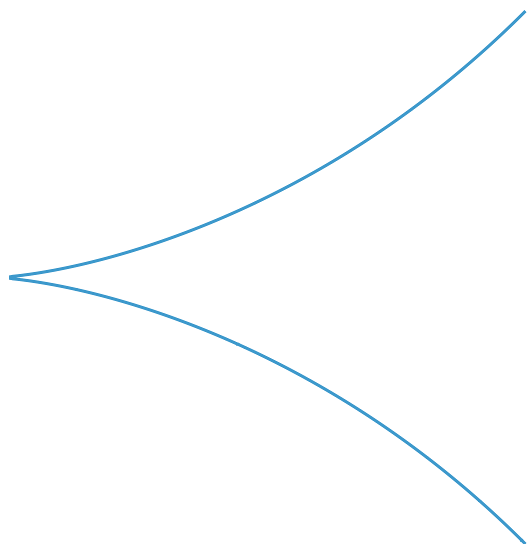

FIGURE 2. The Cissoid of Diocles.

Problem C (10 points): The planar curve

$$r(t) = \begin{bmatrix} \sin(3t) \\ \sin(2t) \end{bmatrix}$$

with $t \in [0, 2\pi)$ is an example of a **Lissajoux figure**.
a) (2 points) Compute the signed curvature function $\kappa(t)$.
b) (2 points) How is the rotation index ρ defined in general? Do not compute the integral but find the index from the picture.
c) (2 points) State the Hopf Umlaufsatz. Does it apply for this curve?
d) (2 points) You are told that the function κ has 6 local maxima. How many vertices does this curve have?
e) (2 points) If a curve has points, where $|r'(t)| = 0$, can you conclude that the curve is **not** arc-length parametrizable?

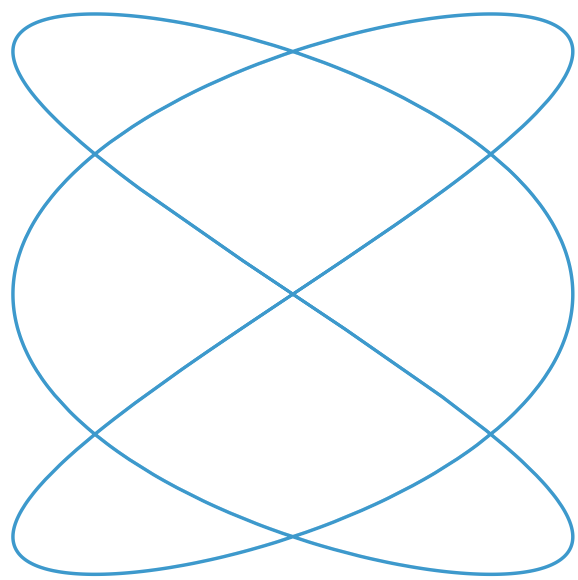

FIGURE 3. Lissajoux figure

Problem D (20 points): Define the surface by the parametrization

$$r(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix} = \begin{bmatrix} u \\ v \\ uv \end{bmatrix} .$$

a) (4 points) Write down the first fundamental form $g = I$.
b) (4 points) Write down the second fundamental form $h = II$.
c) (4 points) Write down the third fundamental form $e = III$.
d) (2 points) How is the shape operator A defined from I, II (in general)?
e) (2 points) How are K and H obtained, if you know I, II, III (in general)?
f) (4 points) We know $e - 2Hh + Kg = 0$ (as you have proven it in a homework). Check this matrix identity in the present case.
Note: In d),e) just state the general definitions without actually computing it for $z = xy$. As for f), an oracle tells you that with $B = 1/\sqrt{1+u^2+v^2}$ curvatures are $\boxed{K = -B^4}$ and $\boxed{H = uvB^3}$. There is no need to verify these two boxed formulas for K, H. The quantity B can be useful in parts a),b),c) using for example $B_u = -uB^3, B_v = -vB^3$.

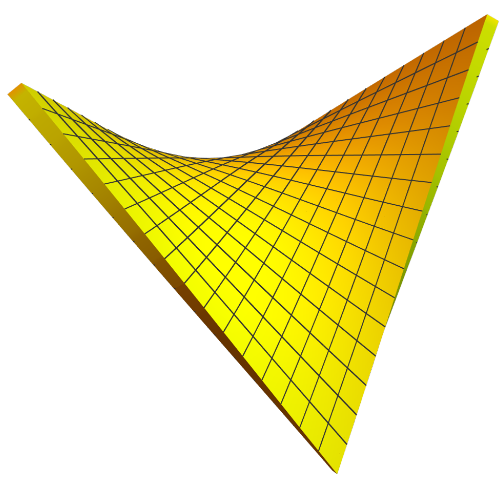

Figure 4. The hyperbolic paraboloid.