DIFFERENTIAL GEOMETRY

MATH 136

To Lecture 21a: Discrete Gravity

21.1. Classical general relativity is based on two fundamental variational principles: **A)** particles move on geodesics, critical points of a variational problem $\mathcal{L}(c)$ on the set of paths c connecting two points. **B)** The metric tensor g is a critical point of the **Hilbert action** $\mathcal{H}(g) = \int_M S_g \; dV_g$. These are **Einstein manifolds** (M, g). They satisfy the Einstein equations $R - \frac{S}{2}g = 0$, where R is the Ricci tensor and S is scalar curvature. In dimension 2, where $S = 2K$, every compact manifold is Einstein by Gauss-Bonnet. In larger dimensions it implies S is constant. Can we model this in the discrete? One of the many approaches was by Regge in 1960. He used an embedding of the finite structure in Euclidean space and so designed a numerical scheme. Can it be done in finite geometry? Finite mathematics is motivated by the possibility that there is no Euclidean space after all. This would happen for example if ZFC was inconsistent. It is well possible that already the real line, the 1-dimensional Euclidean space, is just a pipe dream.

21.2. Replace a Riemannian manifold (M, g) with a discrete manifold $G = (V, E)$ of dimension m. There is no ambient Euclidean space. An edge E is just a pair of vertices. No additional structure like length or distances is assumed. It is just a graph. A naive attempt for part A) is to look at paths (sequences of points in V with adjacent pairs in E) in G and pick the one with the least amount of edges. A path is **simple** if its induced graph is a path graph. This is problematic already in the smallest possible cases. Take for example the icosahedron, a 2-sphere that is one of the smallest 2-dimensional manifolds. There are vertices A,B of distance 2 for which one has already two shortest connection. Even after many Barycentric refinements, we still have always points in distance 2 apart which feature two different geodesics connecting them. We can also not continue a path naturally, something which a geodesic flow should be able to do. Take an edge reaching a vertex of degree 5 in the icosahedron. We would have artificially to stablish a rule on each vertex telling on how an incoming path gets continued or toss a coin. Ugly!

21.3. A better way for part A) is to take maximal ordered simplex $x = (x_0, \dots, x_m)$ and map it into a new simplex $y = (x_1, \dots, x_m, x_0')$, where x_0' is the second point in the 0-sphere $\bigcap_{k=1}^m S(x_k) = \{x_0, x_0'\}$. Call this $y = T(x)$. We can continue this process and define $T^2(x) = T(y)$. This is a deterministic. We stack simplices onto each other. It produces a nice dynamical system and an **exponential map**. Having geodesics with exponential map is crucial to define sectional curvature and so to **Ricci curvature** R and so **scalar curvature** S. Again, all this needs to be done without

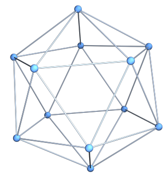

FIGURE 1. Defining a geodesic flow on an icosahedron is problematic if geodesics are paths on the vertex set V. Better: move on triangles.

geometric realization of the structure. The ZFC axiom system might be consistent, forcing mathematics to retrench to the finite.

Theorem 1. *For any m-manifold G, there exists a geodesic flow as a permutation on the finite set P of ordered maximal complete subgraphs of G.*

21.4. We have seen in class that there is for any graph a curvature $K(v)$ that satisfies the Gauss-Bonnet formula $\sum_{v \in V} K(v) = \chi(G)$. In two dimensions, where $K(v) = 1 - \deg(v)/6$, this already encodes the curvature tensor. In higher dimensions, we need to make sense of the Riemann tensor, the Ricci tensor and the scalar curvature. Classically, these curvatures are completely determined from **sectional curvatures**. We therefore need to give a **reasonable notion of sectional curvature** in the finite. The key is that sectional curvature is defined with a reasonable geodesic flow.

21.5. Here is how to define sectional curvature in a m-manifold with $m \geq 2$. Take a triangle $t = (x_1, x_2, x_3) \subset x$ in a maximal ordered simplex x. Every of its edges defines a bone. They are $b_3 = x - (x_1, x_2), b_1 = x - (x_2, x_3)$ and $b_2 = x - (x_1, x_3)$. The dual simplices $(\bigcap_{j=0}^{m-2} S(x_j))$ are cyclic graphs in the dual graph (with maximal simplices are vertices and pairing two if they intersect in a $(m-1)$-simplex). (For $m = 2$, vertices are bones and the unit spheres are the dual bones, for $m = 3$ the edgers are bones and the tetrahedra hinging on it are the dual bone). The bones encode how x is connected to three more maximal simplices y_1, y_2, y_3. Two adjacent bones define new triangle in each of the y_i, allowing to continue the surface. This allows to get sectional curvature for any pair (t, x) where t is a triangle and x is an ordered maximal simplex.

21.6. We can now define Ricci curvature of an edge e of the manifold as the average over all sectional curvatures of triangles t containing e. The scalar curvature S at a vertex v finally is the average over all Ricci curvatures of edges containing v. What property does correspond to the Einstein equations in the continuum? A natural guess is to find a geometry for which a suitably scaled $H = \sum_{v \in V} S(v)$ is maximal or minimal (or to take a short-cut, where $S(v)$ is constant). Looking at such variational problems is pretty unexplored. They could be called the **"Sarumpaet rules"**. [1] One can try different things like minimizing a suitably scaled Hilbert action H. Any local modification of the q-manifold would keep the Hilbert action or make it larger. Encouraging is that any 2-manifold is an Einstein manifold in this sense, because Gauss-Bonnet assures that the Hilbert action is constant for all 2-manifolds of the same type. In general, look for structures with constant S.

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[1] Greg Egan novel "Schild's ladder". Schild's ladder is a numerical scheme to compute geodesics.