Lecture 20: The Curvature Tensor

19.1. While $\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}]$ is not a tensor, the **Riemann curvature**

$$R^s_{ikj} = \frac{\partial}{\partial u^k}\Gamma^s_{ij} - \frac{\partial}{\partial u^j}\Gamma^s_{ik} + \sum_r \Gamma^r_{ij}\Gamma^s_{rk} - \sum_r \Gamma^r_{ik}\Gamma^s_{rj}$$

is a $(1,3)$ tensor. The skew-symmetric R^s_i generates an orthogonal change when rotating in the k,j plane. We also define R in the form of the $(0,4)$-tensor $R_{mikj} = \sum_s g_{ms} R^s_{ijk}$.

19.2. Without coordinates: let $X = \sum_i X^i e_i$ and $Y = \sum_i Y^i e_i$ denote **vector fields** = $(1,0)$ tensor fields. [1] The notation $e_i = \partial_{u^i}$ reflects that a vector field $X = \sum_i X^i e_i$ also defines a linear map on functions $Xf = \sum_i X^i f_{u^i} = df X$. It is also known as **Lie derivative**. [2] Since every vector field X also is a linear map, one can look at the commutator $[X,Y] = XY - YX$, which is by Leibniz again a vector field. Proof: in coordinates $X = \sum_j X^j e_j, Y^j = \sum_j Y^j e_j$, this **Lie bracket** is $[X,Y]^i = \sum_j X^j \partial_j Y^i - Y^j \partial_j X^i$.

19.3. The **covariant derivative** $\nabla_X Y$ is a new vector field. Axiomatically it is determined by **Leibniz** $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, **metric compatibility** $g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y,Z)$, and being **torsion free** $\nabla_X Y - \nabla_Y X = [X,Y]$. The **fundamental theorem of Riemannian geometry** assures that there exactly one such derivative: it is $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$ and it determines the Riemann curvature tensor R.

19.4. The curvature tensor now also can be written as $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ so that $R = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. We have $g(e_m, R(e_k, e_j)e_i) = R_{mikj}$. Intuitively, $A = R(e_k, e_j)$ is the Frenet matrix when parallel transporting along a small rectangular loop spanned by e_k, e_j. The $A^s_i \in so(q, \mathbb{R})$ defines then a rotation in $SO(q, \mathbb{R})$ when doing the loop.

19.5. For linearly independent vectors u, v, the **sectional curvature**

$$K(u,v) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2}$$

is an intuitive approach to curvature. It only depends on the plane spanned by u,v and not the coordinate system. It only depends on the plane defined by the tangent vectors u,v. If M is two dimensional, it agrees with the Gauss curvature. But this is not obvious as it includes the Theorema egregium!

[1] The literature uses capital letters for vector fields, $\sum_i X^i e_i$ rather than $\sum_i v^i e_i$.

[2] or directional derivative $\nabla f \cdot v$ in multi-variable calculus, if $|v| = 1$.

Theorem 1. *The sectional curvatures determine the Riemann curvature tensor.*

19.6. The **Ricci curvature** R is a contraction of the curvature tensor $\boxed{R_{ik} = \sum_j R^j_{ijk}}$.

The **scalar curvature** is then the contraction of the Ricci curvature $\boxed{S = \sum_{j,k} g^{jk} R_{jk}}$. In two dimensions, it is twice the Gauss curvature. The **Einstein tensor** G is defined as $\boxed{R - Sg/2}$. A metric is called an **Einstein metric** if $R = \lambda g$ for some λ. Define the **Hilbert functional** $S(g) = \int_M S_g dV_g$ and the inner product on $(0,2)$ tensors as $\langle a, b\rangle_g = \int_M \sum_{i,j} a(e_i,e_j)b(e_i,e_j)\ dV$. Under which conditions is Hilbert functional extremal? [3]

Theorem 2. $\frac{d}{dt}S(g+th) = \langle Sg/2 - R), h\rangle_g$.

Theorem 3. *Every 2-manifold is an Einstein manifold:* $Sg/2 - R = 0$.

Proof. The reason is that $K = S/2$ and that the Hilbert functional $S(g) = 2\int_M K dV = 4\pi\chi(M)$ does not depend on the metric by Gauss-Bonnet, so that every g is a critical point. □

We see that for $\dim(M) = 2$, the Ricci tensor R is K times the Riemannian metric tensor g. This is not obvious as it leads to the Theorema egregium.

19.7. To prepare for relativity, generalize Riemannian manifolds. A **metric tensor** on a linear space E is a symmetric $(0,2)$ tensor which is **non-degenerate** that is $g(u,v) = 0, \forall v \in E \Rightarrow u = 0$. A **metric tensor field** g is a tensor field $g \in T^0_2(M)$ such that $g(x)$ is a metric tensor in $T^0_2(T_xM)$. This means that for any vector fields X, Y the function $x \to g(x)(X(x), Y(x))$ is smooth. A **pseudo Riemannian manifold** (M, g) is a smooth manifold with a metric tensor field g on M. It is a **Riemannian manifold**, if g is positive definite, meaning $g(x)(v,v) \geq 0$ for all v. The **length** of a vector $v \in T_pM$ is defined as $||v|| = \sqrt{|g(p)(v,v)|}$, where $g(p)(u,v) = \sum_{ij} g_{ij}(p)u^iv^j$. [4] A vector in T_pM of length zero is called **null**. Vectors u for which $g(p)(u,u) = \sum_{ij} g_{ij}u^iu^j < 0$ are **time like**, vectors u with $g(p)(u,u) = \sum_{ij} g_{ij}u^iu^j > 0$ **space like**. The **length** of the curve is defined by $\int_a^b ||\dot{x}(t)||\ dt$.

19.8. Which signatures can be realized?

Theorem 4. *On any smooth manifold there exists a Riemannian metric* g.

Proof. We need to show that there exists $g \in T^0_2(M)$ which is symmetric, non-degenerate and positive definite: let $\{U_i, \phi_i\}$ be an atlas for M and let p_i be a **partition of unity**, subordinate to the cover U_i. Let q be a Riemannian metric on $\mathbb{R}^n$. For example $[q] = \text{Diag}(1,1,1\ldots,1)$. Let $q_i = \phi_i^* q$ be the pull back metrics on U_i. Define $g(p) = \sum_i g_i(p)q_i(p)$. This is smooth and positive definite because for $p \in M$ and u in the tangent space T_pM, we have $g(p)(u,u) = \sum_i g_i q_i(u,u) > 0$. □

19.9. It is not always possible to build on a given manifold a metric of a given signature. For example, on the sphere $M = S^2$, there exists **no Lorentzian metric**, that is a metric of signature $(-1,1)$. The reason is that one can not comb a 2-sphere.

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2025

[3] A proof can be found on pages 312-320 in Kuehnel.

[4] Mind the absolute value here!