

DIFFERENTIAL GEOMETRY

MATH 136

To Lecture 18: What is a manifold?

18.1. We measure distances using light or sound and so observe to live in a **metric space** M , a topological space in which open sets are defined by small open balls $B_r(p)$ defined by the metric. Empirically we also observe our physical space M is a **3-manifold**. A sphere $S_r(p)$ in M given as the set of points in positive distance r from a point is a 2-sphere. Any of these objects $N = S_r(p)$ themselves are a 2-manifold: it has the property that $S_r(p)$ in N is a circle for small r . A circle is a 1-manifold: a small sphere $S(r)$ in a circle is a 0-sphere, a discrete set of 2 points. Every discrete set of points is a 0-manifold because $S_r = \emptyset$ and \emptyset is declared to be a **(-1)-sphere**. The notion of “manifold” is so inductively linked to the notion of “sphere”. The induction is with respect to dimension.

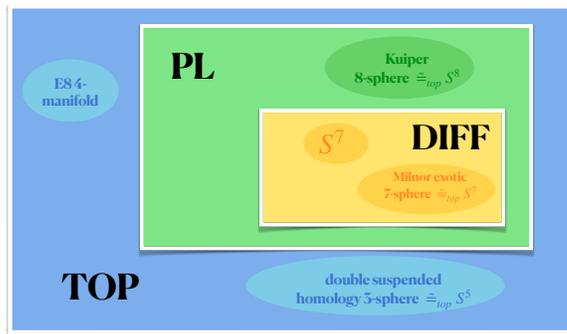


FIGURE 1. Three categories of manifolds.

18.2. The question “**what is a sphere?**” was first studied by Herman Weyl. Topology tells us that spheres are connected for dimension $q > 0$ and simply connected for $q > 1$. The **standard q -sphere** $S_r(x) = \{x \in \mathbb{R}^{q+1}, \sum_{i=1}^{q+1} x_i^2 = 1\}$ defined in Euclidean space is a q -manifold of Euler characteristic $1 + (-1)^q$. It has an atlas with 2 patches given by stereographic projections $\phi_{\pm}(x) = (x_1, \dots, x_q)/(1 \pm x_{q+1})$. A **sphere** is a manifold which can be covered with 2 contractible subsets. It also admits a Morse function with exactly two critical points. These properties are equivalent and each forces M to be a sphere. Can we define a sphere without Euclidean baggage? We definitely want the Euler characteristic to be $1 + (-1)^q$. For a “Dehn-Sommerville q -manifolds”, we only need to ask that $S_r(p)$ is a “Dehn-Sommerville $(q-1)$ manifold” of dimension $(q-1)$ and Euler characteristic $1 - (-1)^q$ if r is small enough. The Dupin Cyclide from Homework 1 is such a surface.

18.3. The standard definition of a q -manifold is to define it as a paracompact Hausdorff space that is locally homeomorphic to an open ball in Euclidean space. This leads to the category of **topological manifolds** TOP . Not all topological manifolds can be triangulated. One has therefore also introduced the class PL of **piecewise linear manifolds** that admit a triangulation. Not all triangulated manifolds admit a differentiable structure; others admit many different differentiable structures. The subclass $DIFF$ of **differentiable manifolds** now produces a set of inclusions $DIFF \subset PL \subset TOP$ where each is strict. Kervaire constructed in 1960 a manifold in $PL \setminus DIFF$, the join of a Poincaré homology sphere with a circle is in $TOP \setminus PL$ (Edwards and Cannon 1970-79). Milnor constructed in 1959 inequivalent spheres in $DIFF$. Taubes 1987 got uncountably many differentiable structures in \mathbb{R}^4 .

18.4. When we talk about discrete manifold we mean PL manifolds as any discrete manifold can be geometrically realized as a PL manifold. It is no surprise therefore that in differential geometry, where we are mostly interested in $DIFF$, we can do essentially everything using discrete manifolds.

18.5. The three categories can also be extended to the Dehn-Sommerville case. Some Dehn-Sommerville spaces are algebraic varieties given as the intersection of polynomial equations. But in general they fall into the category of **schemes**. We do not need to get fancy however: here is a reasonable definition:

A **Dehn-Sommerville q -manifold** M is a compact metric space such that for every point $p \in M$ there exists $\rho > 0$ such that every sphere $S_r(p)$ is a Dehn-Sommerville $(q-1)$ -manifold of Euler characteristic $1 - (-1)^q$ for $0 < r < \rho$. To start induction, declare \emptyset to be a Dehn-Sommerville (-1) -manifold.

18.6. Note that the just given definition does not use any continuous map and so is extremely elegant. And here is a definition to define a topological manifold without the use of charts. It immediately goes over to combinatorial settings, where no reference to Euclidean spaces is needed any more.

A metric space (M, d) is called **contractible** if there exists a point $p \in M$ and a continuous map $F(s, x) : [0, 1] \times M \rightarrow M$ such that $F(0, x) = x$ and $F(1, x) = p$ for all x . A **q -manifold** M is a compact metric space such that for every $p \in M$ there exists $\rho > 0$ such that every $S_r(p)$ is a $(q - 1)$ -sphere for $0 < r < \rho$. A **q -sphere** M is a q -manifold for which there exists $p \in M$ such that $M \setminus \{p\}$ is contractible. The empty set \emptyset is declared to be the unique (-1) -sphere.

18.7. The just given definition produces the class TOP of compact topological manifolds in finite dimensions. The assumption to have a metric space rather than a paracompact topological space is no a loss of generality in finite dimensions because of the Whitney embedding theorem. The definition produces for every point p a neighborhood chart. One can prove by induction that the ball $B_r(p) = \{x \in M, d(x, p) < r\}$ is homeomorphic to the **standard unit ball** in \mathbb{R}^q . Also the reverse holds because a compact $M \in TOP$ is known to be a q -sphere, if $M \setminus \{p\}$ is contractible for some $p \in M$ (actually for all $p \in M$).