DIFFERENTIAL GEOMETRY

MATH 136

# Lecture 17: Global Gauss-Bonnet

**17.1.** Now we are ready to prove the **global Gauss-Bonnet theorem** for a 2-manifold $M$ without boundary. The surface $M$ is triangulated by a **discrete manifold** $G = (V, E, F)$, where the faces $F$ are the triangles defined by the graph $(V, E)$. The discrete manifold is geometrically realized in $M$ as a collection of points, a collection of curves connecting vertices. The geometrically realized network divides $M$ up into triangular faces $M_i = r(U_i)$. The Euler characteristic of $M$ is $\chi(M) = V - E + F$. As we have seen, $\chi(M)$ does not depend on the triangulation: topological changes like removing a disc and gluing in a new disc (we called this as a **connected sum** $M \to M\#S^2$) or doing a **Barycentric refinement** does not change $\chi(M)$.

**Theorem 1** (Gauss-Bonnet theorem)**.** *For a compact 2-manifold,* $\iint_M K dV = 2\pi\chi(M)$.

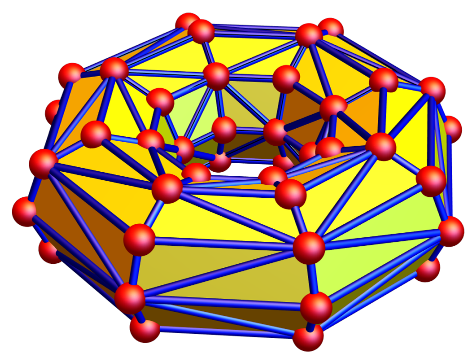

Figure 1. A triangulated manifold $M$. We apply the local Gauss-Bonnet theorem on each of the triangles. The edge contributions produce with Green's theorem the face curvatures $\iint_{M_i} K \; dV$ as well as $2\pi F$. The vertex contributions produce, using the Euler Handshake lemma, $2\pi(V - E)$. Overall, we have $2\pi(V - E + F) = 2\pi\chi(M)$. The picture shows a $M = \mathbb{T}^2$ with $V = 64$ vertices.

**17.2.** We will use the local Gauss-Bonnet theorem for each triangle $U_i$ with angles $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$. We first of all want to understand what happens if we glue together triangles such that the frame field $X = [zw_u, zw_v]$ can be define on the union. On the curve obtained by intersecting two adjacent triangles, the line integral of $X$ cancels.

**Lemma 1** (Cancellation). *If two triangles $M_1, M_2$ meet in a curve $C$ and $C_1, C_2$ are the parametrizations matching the $M_1, M_2$, then $\int_{C_1} X\, dr + \int_{C_2} X\, dr = 0$.*

*Proof.* If $X$ is a 1-form and $C$ is a curve and $-C$ is the curve passed backwards, then $\int_C X\, dr + \int_{-C} X\, dr$. This is what happens here. You can see the identity also as a consequence of Green also, noting that $C \cup -C$ encloses an "empty region". □

**17.3.** We see that all the 1-form contributions from the edges are zero. The contributions from the faces $M_i$ add up:

**Lemma 2** (Additivity). $\int_M K\, dV = \sum_i \int_{M_i} K\, dV$.

*Proof.* The patches $M_i$ are all disjoint. Their union is $\bigcup_i M_i = M$. Areas of disjoint regions add up. □

**17.4.** The contributions from the vertex degrees $d_i = |S(v_i)|$ add up too. [1]

**Lemma 3** (Euler handshake). *If $(V, E)$ has vertex degrees $d_i$, then $2E = \sum_i d_i$.*

*Proof.* You prove this in a homework. □

**17.5.** We still have to look at the contributions from the vertices. At each point $P_i$ we have angles $\alpha_{ij}$ for $j = 1, \dots, d_j$, where $d_j$ is the vertex degree.

**Lemma 4** (Adding vertex curvatures). $\sum_{i=1}^{V} \sum_{j=1}^{d_i} \kappa_{ij} = 2\pi E - 2\pi V$.

*Proof.* (i) $\sum_{i=1}^{V} \sum_{j=1}^{d_i} \kappa_{ij} = \sum_{i=1}^{F} \sum_{j=1}^{3} (\pi - \alpha_{ij})$. (ii) $\sum_{k=1}^{V} \sum_{j=1}^{d_k} \pi = 2\pi E$. (iii) $\sum_{k=1}^{V} \sum_{j=1}^{d_k} \alpha_{kj} = 2\pi V$. □

[2]

**17.6. Proof of the global Gauss-Bonnet theorem**:
The local Gauss-Bonnet theorem told us $\sum_i [\iint_{U_i} K dV + \sum_j \kappa_{ij} - 2\pi] = 0$. This means that $\iint_U K\, dV + \sum_{i,j} \kappa_{ij} - 2\pi F = 0$. Therefore, using the previous lemma:

$$\iint_U K\, dV = 2\pi V - 2\pi E + 2\pi F = 2\pi \chi(M)\ .$$

**17.7. 1)** If M is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, then $\iint_M K\, dV = 4\pi$ (HW).
**2)** A genus $k$ surface has $\iint_M K\, dV = 2\pi(2 - 2k)$. For a torus $\iint_M K\, dV = 0$.
**3)** A Klein bottle is obtained by gluing two Möbius strips together. $\iint_M K\, dV = 0$. Because each Möbius strip has Euler characteristic 0 (you computed that in an example), and the Möbius strip can be realized so that the boundary curvature $\chi_g$ is zero.

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[1]If $V, E, F$ are the vertices, edges and faces. It is custom to write its cardinalities as $V, E, F$.

[2]See youtube video. This year I fell live (but not "free solo"). Some toe or thum acrobatics: the angles $\alpha_{ij}$ for the triangles $F_i$ are relabeled $\beta_{ij}$ for vertices. Also use the karate kick $3F = 2E$.