DIFFERENTIAL GEOMETRY

MATH 136

Lecture 16: Local Gauss-Bonnet

16.1. We now prove the Gauss-Bonnet theorem in the situation when $U \subset R$ is a polygon. The parametrization $r : R \to M$ plants the polygon $r(U) \subset r(R)$ into the surface M.

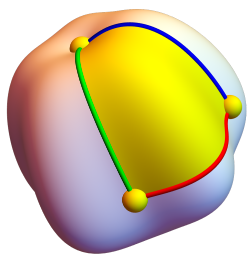

FIGURE 1. The local Gauss-Bonnet theorem tells that face, edge and vertex curvatures of a polygon $r(U)$ in a manifold M add up to 2π.

16.2. A **simple polygon** in M is the image $r(U)$ of a simple polygon $U \subset \mathbb{R}^2$ such that r is smooth and injective on U. Its **Euler characteristic** is $\chi(U) = |V| - |E| + |F| = 3 - 3 + 1 = 1$. As in the discrete Hopf Umlaufsatz, the vertex curvatures are defined as $\kappa_i = \pi - \alpha_i$, where α_i are the polygon angles. The **angles α_i of the polygon** are defined by $\cos(\alpha_i) = \dot{x}_i(1) \cdot \dot{x}_{i+1}(0)$, the dot product of the velocity vectors of the arcs at the end of the incoming and the beginning of the outgoing arc.

16.3. Let U be a simple polygon on M. There are three contributions to curvature: the **face curvature** is the integral of K over the interior, the **geodesic curvature** integrates sectional curvature κ_g over the edges C_j and then there are the **vertex curvatures** $\kappa_j = \pi - \alpha_j$ attached to the vertices.

Theorem 1 (Local Gauss-Bonnet). $\iint_U K dV + \sum_j \int_{C_j} \kappa_g(x_j(t))\, dt + \sum_j \kappa_j = 2\pi$.

16.4. If $x(t) = r(u(t), v(t))$ parametrizes the boundary of the surface $M = r(U)$, we can assume that it is parametrized by arc-length. The velocity vector $\dot{x}$ is a 3-vector tangent to the surface. We look at the orthonormal frame field (z, w) from last time. The **geodesic curvature** of a curve x is defined at points where x is smooth and given as $\boxed{\kappa_g = (n \times \dot{x}) \cdot \ddot{x}}$. Unlike $\kappa = |\dot{x} \times \ddot{x}|$, it is signed. So is the **normal curvature** $\boxed{\kappa_n = n \cdot \ddot{x}}$. Since $\dot{x} \cdot \ddot{x} = 0$, Pythagoras gives $\kappa_g^2 + \kappa_n^2 = \kappa^2$. The velocity vector of the curve can be expressed as an angle so that $\boxed{\dot{x} = \cos(\theta)z + \sin(\theta)w}$. We write $\dot{w}$ for $\frac{d}{dt}w(x(t))$.

Lemma 1 (Geodesic lemma). *$\kappa_g = \dot{\theta} - (z \cdot \dot{w})$.*

Proof. Fill in the parts of the definition $\kappa_g = (n \times \dot{x}) \cdot \ddot{x}$:
(i) $n \times \dot{x} = \cos(\theta)w - \sin(\theta)z$.
(ii) $\ddot{x} = \dot{\theta}(-\sin(\theta)z + \cos(\theta)w) + \cos(\theta)\dot{z} + \sin(\theta)\dot{w}$.
(iii) So, $\kappa_g = (n \times \dot{x}) \cdot \ddot{x} = \dot{\theta} - z \cdot \dot{w}$ □

16.5. We can now prove the local Gauss-Bonnet theorem:

Proof. (i) Integrating the geodesic lemma gives

$$\int_0^L \kappa_g \, dt = \int_0^L \dot{\theta} \, dt - \int X \, dr$$

(ii) Green's theorem assures that $\int X \, dr = \iint_U K \, dV$ as $KdV = dX$.
(iii) The Hopf Umlaufsatz for curved polygons gives $\int_0^L \dot{\theta}(t) \, dt + \sum_j (\pi - \alpha_j) = 2\pi$.
(iv) Putting (i),(ii),(iii) together gives the proof. □

16.6. Example 1) If K is constant 0 and U is a triangle, Gauss Bonnet is $\kappa_1+\kappa_2+\kappa_3 = 2\pi$. This is equivalent to $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ you know from elementary school geometry. For an **n-gon**, the identity $\sum_{i=1}^n \kappa_i = 2\pi$ is equivalent to $\sum_{i=1}^n \alpha_i = (n-2)\pi$.

16.7. Example 2) If $M = \mathbb{S}^2$ is a sphere of radius 1, then curvature is $K = 1$. The integral $\iint_U K \, dV$ is the **area** $|U|$ **of the triangle**. The formula becomes $|U| + \sum(\pi - \alpha_i) = 2\pi$ and so $\boxed{\alpha_1 + \alpha_2 + \alpha_3 = |r(U)| + \pi}$. This is **Girard's theorem** or **Harriot's theorem** in spherical geometry, named after Albert Girard or Thomas Harriot.

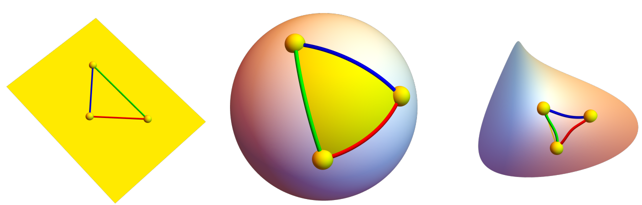

Figure 2. A triangle in the plane has $\alpha_1+\alpha_2+\alpha_3 = \pi$. For a spherical triangle of area A, Harriet's theorem gives $\alpha_1 + \alpha_2 + \alpha_3 = \pi + A$. On a hyperbolic space, Lambert's theorem is $\alpha_1 + \alpha_2 + \alpha_3 = \pi - A$.

16.8. Example 3) If M is a surface of constant curvature -1, a triangle is called **hyperbolic** Now, $\iint_U KdV = -|U|$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi - |U|$, a formula found by Johann Heinrich Lambert. The right hand side $\pi - |U|$ is called **spherical defect**.

16.9. Example 4) Take a sphere with a simple closed geodesic on it, integral of K on each half is 2π. The total integral is 4π.

16.10. Example 5) If $K = 0$ and $r(U)$ is a region in the plane bound by a simple smooth curve, we have the **Hopf Umlaufsatz** $\int \kappa_g(t) \, dt = 2\pi$.