

DIFFERENTIAL GEOMETRY

MATH 136

Lecture 13: The exponential map

13.1. The geodesic differential equation $\ddot{x}^k + \sum_{i,j} \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$ can be written as a first order system $\frac{d}{dt}[x, \dot{x}] = [\dot{x}, f(x, \dot{x})]$ if the first fundamental form g is twice differentiable. This ordinary differential equation therefore has **local solutions** for some time $t \in (-a, a)$ by the Piccard existence theorem. But the solutions exist for all time. No “blow up” is possible if the surface is smooth, regular and closed. The reason is the following:

Lemma 1. *If $x(t)$ is geodesic, then $\langle \dot{x}, \dot{x} \rangle = \sum_{i,j} g_{ij} \dot{x}^i \dot{x}^j$ is preserved.*

Proof. The Hamiltonian $H = -F + \sum_j \dot{x}^j F_{\dot{x}^j}$ is also called the Legendre transform. We get $\frac{d}{dt}H = -\sum_j F_{x^j} \dot{x}^j - \sum_j F_{\dot{x}^j} \ddot{x}^j + \sum_j \ddot{x}^j F_{\dot{x}^j} + \sum_j \dot{x}^j \frac{d}{dt}F_{\dot{x}^j} = 0$. Use Euler-Lagrange to replace the last term with $\sum_j \dot{x}^j F_{x^j}$. For $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle$, we have $H = -F + 2 \sum_j \dot{x}^j \dot{x}^j = -F + 2F = F$. We see $H = F = \langle \dot{x}, \dot{x} \rangle$ is preserved energy. \square

Theorem 1 (Hopf-Rynov). *For regular, compact, smooth M , geodesics exist globally.*

Proof. If M is C^4 then Γ is differentiable. The Piccard existence theorem gives local solutions in the unit tangent bundle $(p, v) \in M \times S^{m-1}$. A regular compact manifold is complete in the sense that all Cauchy sequences have limits. The only way that a solution path could not be continued is that $\dot{x}(t)$ blows up. Otherwise, we could restart the differential equation at the end point a of a maximal interval $(-a, a)$ of existence. By the lemma, a blow up of $\dot{x}(t)$ is not possible. \square

13.2. Remarks: **a)** The regularity is necessary. On a piece-wise smooth manifold like a cube, a geodesic hitting a corner can not be continued continuously. **b)** There are compact Lorentzian manifolds like the Clifton-Pohl torus that are not complete. **c)** The lemma is important. The proof shows that each variational problem gets with a **Legendre transform** $H = -F + \sum_j \dot{x}^j F_{\dot{x}^j}$ to an “energy” H that is preserved. ¹

13.3. The **exponential map** $\exp_p : T_p M \rightarrow M$ is obtained by defining $\exp_p(0) = p$ and for $v \neq 0$, define $\exp_p(v)$ by taking $v/|v|$ as initial direction of the geodesic flow and evolving it for time $|v|$. The image $\exp(S_r(0)) = W_r(p)$ is called the **wave front**. It is the set of all points which can be reached from p by running from it a geodesic of length p . Wave fronts are **geodesic circles** for small t but in general become very complicated.

¹For more, see J. Moser, Selected Topics in the Calculus of Variations. (Notes by O. Knill) 2002

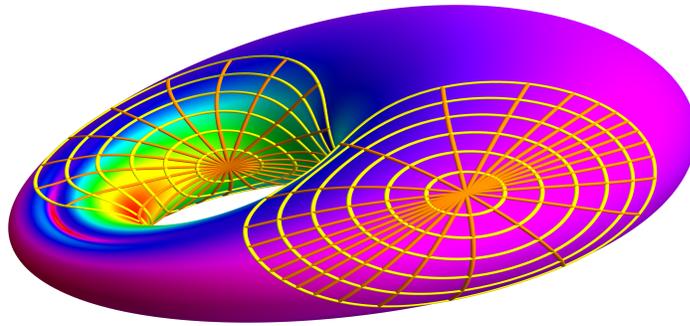


FIGURE 1. The exponential map evolves all possible geodesics from p . If all these geodesics are stopped at time t , we get a wave front $W_t(p)$.

13.4. The **radius of injectivity** of M is the smallest r such that the exponential map $B_r \subset T_p M \rightarrow M$ is injective.

Lemma 2. For a compact manifold M , the radius of injectivity is positive.

Proof. When fixing a point p , there is a $B_r(0) \subset \mathbb{R}^m$ such that that \exp_p is invertible. This follows from the **inverse function theorem** and the fact that $d\exp_p = 1$ (identity matrix) at p because $\exp_p(v) - v = O(|v|^2)$ by definition. Let $r(p)$ be maximal radius on which $\exp_p(B_r(0))$ is differentiable. This function $r(p)$ is continuous in p and positive. By compactness of M and the **extremal value theorem**, there is a minimum, a lower bound. \square

13.5. For fixed p , critical values of \exp_p form the **caustic** of p . If r is the radius of injectivity, the open set $U = \exp_p(B_r(0)) \subset M$ is called the **normal neighborhood** of p . Lets look at the two dimensional case:

Lemma 3. On U there are coordinates (ρ, θ) such that $g = I = \begin{bmatrix} 1 & 0 \\ 0 & G \end{bmatrix}$ satisfying $\lim_{\rho \rightarrow 0} G(\rho, \theta) = 1$.

Proof. These are called **geodesic polar coordinates** because they come from the exponential map. Since velocity is preserved, the radial direction does not expand. \square

13.6. This implies:

Theorem 2 (Gauss Lemma). For every unit vector v , the radial geodesics $\{\exp_p(sv), s \leq t\}$ is normal to the wave front $W_t(p)$.

Proof. Within $U = \exp_p(B_r(0))$ this is clear by the coordinates. \square

13.7. Remarks. 1) Geodesic coordinates with $I = g = \text{diag}(1, g_{22} \dots, g_{mm})$ exist on any m -manifolds. 2) For 2-manifolds, linearising the geodesic flow affects only the vector perpendicular to the geodesic $x(t)$. This is called a **Jacobi field**. For surfaces, for fixed p and v , we get a **Jacobi differential equation** $z'' = -K(x(t))z$, where $z(t) = G(x(t))$ in the normal patch. The roots of $z(t)$ belong to caustic points $\exp_t(p)$.