DIFFERENTIAL GEOMETRY

MATH 136

Lecture 12: Geodesics

12.1. If $M = r(R)$ is a regular manifold, define the space X of regular paths $x(t)$ that start at $x(a) \in R$ and end at $x(b) \in R$. If $F(x, \dot{x})$ is a function of position x and velocity $\dot{x}$, we can minimize $E(x) = \int_a^b F(x, \dot{x})\, dt$ by looking for paths $x(t)$ at which the variation is zero. [1]

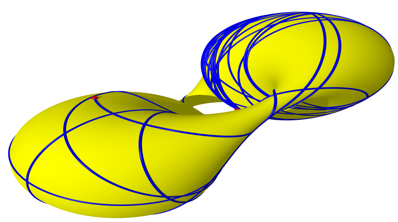

FIGURE 1. This geodesic on a torus was computed in Povray with the Schield's ladder method: evolve freely in $\mathbb{R}^3$ but stay glued to the surface.

Theorem 1 (Euler-Lagrange)**.** *If x minimizes E, then* $\boxed{F_x(x, \dot{x}) = \frac{d}{dt} F_{\dot{x}}(x, \dot{x})}$.

Proof. For a minimum, the change $E(x + \xi) - E(x)$ of a displacement $x + \xi$ of x satisfies $\int_a^b F(x + \xi, \dot{x} + \dot{\xi}) - F(x, \dot{x})\, dt \geq 0$. As Fermat knew, we better have $dE\xi = \lim_{h \to 0}(E(x + h\xi) - E(x))/h = 0$ because a non-zero limit would make $E(x + h\xi)$ larger or smaller than $E(x)$ for small h. By the chain rule, $dE\xi = \int_a^b F_x(x, \dot{x})\xi + F_{\dot{x}}(x, \dot{x})\dot{\xi}\, dt$. Integration by parts, using $\xi(a) = \xi(b) = 0$, gives $dE\xi = \int_a^b [F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x}]\xi(t)\, dt$. In order that this is zero for all ξ, we better have $[F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x})] = 0$ for all $t \in [a, b]$. Proof. If $\neq 0$ at some point $t \in [a, b]$, it would be non-zero in a neighborhood U of t, allowing to find a smooth function ξ that is positive in U and 0 else, producing a nonzero change $dE\xi$. □

12.2. To understand minima if $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle = g(x)(\dot{x}, \dot{x}) = \sum_{i,j} g_{ij}(x)\dot{x}^i \dot{x}^j$, we need notation. M was defined as a regular map $r : R \to \mathbb{R}^n$ giving points $r(u^1, \dots, u^m) \in \mathbb{R}^n$. Define the **Christoffel symbols** $\boxed{\Gamma_{ijk} = r_{u^i u^j} \cdot r_{u^k}}$. The product rule gives

$$\partial_{u^k} g_{ij} = r_{u^i u^k} \cdot r_{u^j} + r_{u^i} \cdot r_{u^j u^k} = \Gamma_{ikj} + \Gamma_{jki} \ ,$$

[1] $x(t) = (x^1(t), \dots, x^m(t)) = (u^1(t), \dots u^m(t))$ as most texts use this notation. For typographical reasons, write $\dot{x}^k$ rather than x'^k. If $r(u, v)$ parametrizes M, paths $x(t) = (u(t), v(t)) \in R$ define curves $r(x(t)) \in M$. "Variation" instead of "derivative" avoids confusion with $\dot{x}$. Variations are directional derivatives in an infinite dimensional space X of paths between two fixed points.

$$\partial_{u^i} g_{jk} = r_{u^j u^i} \cdot r_{u^k} + r_{u^j} \cdot r_{u^k u^i} = \Gamma_{jik} + \Gamma_{kij} \ ,$$
$$\partial_{u^j} g_{ki} = r_{u^k u^j} \cdot r_{u^i} + r_{u^k} \cdot r_{u^i u^j} = \Gamma_{kji} + \Gamma_{ijk} \ .$$

Adding the second and third and subtracting the first, using Clairaut $\Gamma_{ijk} = \Gamma_{jik}$, gives $2\Gamma_{ijk}$ on the right hand side. So:

Lemma 1. $\boxed{\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}] \ .}$

Using the notation $g^{ij} = (g^{-1})_{ij}$ and $\Gamma^k_{ij} = \sum_{l=1}^m g^{kl}\Gamma_{ijl}$, we get to the main point: [2]

Theorem 2 (Geodesics). *Minima of the action functional $E(x) = \int_a^b \langle \dot{x}, \dot{x} \rangle \ dt$ satisfy*

$$\boxed{\ddot{x}^k + \sum_{i,j=1}^m \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0.}$$

Proof. To show that Euler-Lagrange for $F(x, \dot{x}) = \sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j$ is $2\sum_j g_{jk}\ddot{x}^j + 2\sum_{i,j} \Gamma_{ijk}\dot{x}^i\dot{x}^j = 0$: use notation $\partial_{x^k} g_{ij} = g_{ij,k}$. First get $F_{\dot{x}^k} = \sum_j g_{kj}\dot{x}^j + \sum_j g_{jk}\dot{x}^j$. Then $\frac{d}{dt}F_{\dot{x}^k} - F_{x^k} = \sum_{j,i} g_{kj,i}\dot{x}^i\dot{x}^j + \sum_j g_{kj}\ddot{x}^j + \sum_{j,i} g_{jk,i}\dot{x}^i\dot{x}^j + \sum_j g_{jk}\ddot{x}^j - \sum_{i,j} g_{ij,k}\dot{x}^i\dot{x}^j$. The 1st, 3rd and 5rd terms add up to $2\sum_{i,j} \Gamma_{ijk}\dot{x}^i\dot{x}^j$. The 2nd and 4th give $2\sum_j g_{jk}\ddot{x}^j$. □

We see that the acceleration of a particle moving on a geodesic is determined by the velocity and "gravitational force" terms Γ which involves changes in the metric. Einstein would interpret these changes in metric as "mass".

12.3. With $G(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} = \sqrt{F}$, we get the **arc length functional**

$$I(r) = \int_a^b ||\dot{x}|| \ dt = \int_a^b \sqrt{\langle \dot{x}(t), \dot{x}(t) \rangle} \ dt = \int_a^b \sqrt{\sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j} \ dt \ .$$

Theorem 3 (Maupertius). *Action and length functionals have the same extrema.*

Proof. Because $\frac{d}{dt}G_{\dot{x}} = \frac{F_{\dot{x}}}{2\sqrt{F}}$ and $G_x = \frac{F_x}{2\sqrt{F}}$, the Euler-Lagrange equations $\frac{d}{dt}G_{\dot{x}} = G_x$ are equivalent to the Euler-Lagrange equations $\frac{d}{dt}F_{\dot{x}}/\sqrt{F} = F_x/\sqrt{F}$. We have used that x was regular, so that $F(x, \dot{x})$ is never zero. □

12.4. If $x(t)$ is an arc length parametrized curve on M, the **normal curvature** is defined as $\boxed{\kappa_n = \ddot{x} \cdot n}$. It is the scalar projection acceleration $\ddot{x}$ onto n. By Cauchy-Schwarz, it is smaller or equal than $\kappa = |\ddot{x}|$. Define the **geodesic curvature** $\kappa_g = \ddot{x} \cdot (n \times \dot{x})$. Pythagoras gives $\kappa_n^2 + \kappa_g^2 = \kappa^2$. Note that both κ_n and κ_g can be signed.

Theorem 4 (Schield's ladder). *Geodesics have zero geodesic curvature.*

Proof. Geodesics minimize arc length $L = \int_t^{t+2h} ||\dot{x}|| \ dt$ between two close points $x(t), x(t+2h)$. If $\kappa_g = \ddot{x} \cdot (\dot{x} \times n) \neq 0$ at some point $x(t)$, there would be a shorter connection between $x(t)$ and $x(t+2h)$ than $x(t), x(t+h), x(t+2h)$ violating minimality. □

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[2] It's musical! ∂_{u^i} is co-variant and u^i is contra-variant. Einstein would write $g_{ij}\dot{x}^i\dot{x}^j$ for $\langle \dot{x}, \dot{x} \rangle$.