DIFFERENTIAL GEOMETRY

MATH 136

To Lecture 11: Manifolds as Particles

11.1. All manifolds are assumed to be 2-dimensional, connected and compact without boundary. They form a **monoid** in which addition is the **connected sum** $A \# B$. It means that $\#$ is an associative operation with a **zero element** 0, the **sphere**. Adding a sphere produces a topologically equivalent manifold. Two discrete 2-manifolds are called **homeomorphic** if there exists a sequence of additions or subtractions of spheres that deforms one into the other.

11.2. There is an obvious **asymmetry** between orientable and non-orientable manifolds, as one can see when looking at the **Euler characteristic** $\chi(M)$ which has the property that the genus (spin) $g(M) = 1 - \chi(M)/2$ is additive $g(A \# B) = g(A) + g(B)$.

Theorem 1 (Classification of orientable 2-manifolds)**.** *Any nonzero orientable 2-manifold M is a sum of $k \geq 1$ tori $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ and $\chi(M) = 2 - 2k$.*

Theorem 2 (Classification of non-orientable 2-manifolds)**.** *Any nonzero non-orientable 2-manifold M is a sum of $k \geq 1$ projective planes $\mathbb{P}^2 \# \cdots \# \mathbb{P}^2$ and $\chi(M) = 2 - k$.*

11.3. While for non-orientable manifolds, $g(M)$ covers the half integers $\mathbb{N}/2$, orientable manifolds $g(M)$ only cover the natural numbers $\mathbb{N}$. The missing hole in the doughnut forces the introduction of "half-tori". It also solves the group completion, sees manifolds as particles and genus as spin.

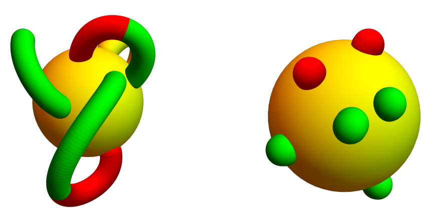

FIGURE 1. A particle with $2g$ Fermions A, B and one with $2g$ Fermions P, Q. Euler characteristic $\chi(G) = 2 - 2g$ relates with spin g.

11.4. One might try to generate more manifolds like $\mathbb{T}^2 \# \mathbb{P}^2$ but this is equivalent to $\mathbb{K}^2 \# \mathbb{P}^2 = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$. One can not embed all manifold in **one single group** because adding a Klein bottle $\mathbb{K}^2$ to a non-orientable M is the same than adding a torus $\mathbb{T}^2$ to it. The **van Dyck's identity** $\mathbb{P}^2 \# \mathbb{K}^2 = \mathbb{P}^2 \# \mathbb{T}^2$ would imply the false identity $\mathbb{K}^2 = \mathbb{T}^2$. We are forced to consider two groups.

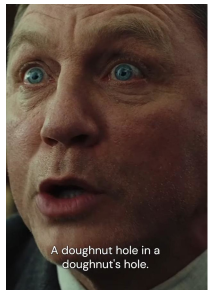

FIGURE 2. *"A doughnut hole in a doughnut's hole! We have to look closer. And when we do, we see the doughnut hole has a hole in its center. "* Inspector Benoit Blanc in "Knives Out".

11.5. If we **symmetrize** we need to introduce **half tori**. In order to Grothendieck complete the monoids to **groups**, we need **anti-projective planes** and **anti-half tori**. Write A for opening a hole and B for opening a second hole and connect it with the already opened hole. Then $A \# B = T^2$. We have naturally extended and completed the monoid to a **finitely presented group** $\langle A, B | A^2 = B^2 = 0 \rangle$. In the non-orientable case, call P the **projective particle** and Q the **anti-projective particle**. Again, the **Pauli principle** postulates, declare P and Q to be involutions. We again get the infinite **dihedral group** $\langle P, Q | P^2 = Q^2 = 0 \rangle$.

11.6. The non-Abelian PQ **group of manifolds** is generated by the projective particle and anti-particle. As Fermions have half spin g, think of P and Q as Fermions.

g=0	0	sphere			
g=1/2	P	projective plane	Q	anti projective plane	g=-1/2
g=1	PQ	Klein bottle	QP	anti Klein bottle	g=-1
g=3/2	PQP	torus with a cap	QPQ	anti torus with a cap	g=-3/2
g=2	$PQPQ$	genus 2 Klein bottle	$QPQP$	anti genus 2 Klein bottle	g=-2

11.7. The non-Abelian AB **group of manifolds** is generated by the torus particle A and torus anti-particle B. Again, one can think of the half spin generators A, B as Fermions.

g=0	0	sphere			
g=1/2	A	half torus	B	anti half torus	g=-1/2
g=1	AB	torus	BA	anti torus	g=-1
g=3/2	ABA	three half torus	BAB	anti three half torus	g=-3/2
g=2	$ABAB$	genus 2 torus	$BABA$	anti genus 2 torus	g=-2

11.8. In both cases, we get the natural **infinite dihedral group** $G = \langle X, Y | X^2 = Y^2 = 1 \rangle$ that is non-Abelian. It is **natural** in the following sense: it can be equipped with a metric such that the group structure on it is determined uniquely by the metric space alone. The integers $\mathbb{Z}$ are not natural because whatever translation invariant metric is picked on $\mathbb{Z}$, it is possible to define both an Abelian and a non-Abelian group structure on it that is group invariant. [1]

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[1]See O. Knill, "On Graphs, Groups and Geometry", 2022