DIFFERENTIAL GEOMETRY

MATH 136

Lecture 11: Discrete Gauss-Bonnet

11.1. To see the Gauss-Bonnet result for a general surface M, we need to define **Euler characteristic** $\chi(M)$. It is defined by a **triangulation** of M. This can well be done in the language **graph theory**, initiated by Leonhard Euler.

11.2. A **graph** $G = (V, E)$ is a finite set V of **vertices** or **nodes** and a finite set E of different **edges** or **connections** (a, b) with $a \neq b$. Every subset V' of V **generates** a subgraph (V', E'), where $E' = \{(a, b) \in E, a \in V', b \in V'\}$. We can so associate a subset V' of V the subgraph it generates. A pair of adjacent vertices for example generates a K_2 subgraph. A pair of non-adjacent vertices generates $S^0 = \overline{K_2}$ the graph with two points and no vertices, which is also known as the 0**-sphere**.

11.3. A circular graph C_n with $n \geq 4$ vertices is called a **circle** of length n. The **unit sphere** $S(v)$ of a vertex v is the subgraph generated by all immediate neighbors of v. A **2-manifold** is a graph for which every unit sphere is a circle. A 2-manifold graph G embedded as a subset $|G| \subset M$ defines a **triangulation** of M; $v \in V$ is realized as a point in M, an edge $e \in E$ is realized as a simple curve in M parametrized by an interval, a connected component in the complement of $|G|$ is regularly parametrized by a triangle $R \subset \mathbb{R}^2$.

11.4. A complete subgraph K_3 of G is also called a **triangle** or a **face** in G. The **Euler characteristic** of a 2-manifold is defined as $\chi(G) = |V| - |E| + |F|$, where $|X|$ is the **cardinality** of X. The **curvature** of a 2-manifold is defined as $\boxed{K(v) = 1 - |S(v)|/6}$. The following theorem goes back to **Victor Eberhard**.

Theorem 1 (Gauss-Bonnet). *For a 2-manifold,* $\sum_{v \in V} K(v) = \chi(G)$.

Proof. Define the function $\omega(x)$ on $X = V \cup E \cup F$ as $\omega(x) = (-1)^{\dim(x)}$ where $\dim(x) = |x| - 1$ is the dimension one less than the number $|x|$ of vertices in x. So, $\chi(G) = |V| - |E| + |F| = \sum_{|x|=1}(-1)^0 + \sum_{|x|=2}(-1)^1 + \sum_{|x|=3}(-1)^2 = \sum_{x \in X} \omega(x)$. If all values -1 from an edge (a, b) are distributed equally to (a, b) and all the values 1 from a face (a, b, c) are distributed equally to the vertices a, b, c, we end up with a function K that is only non-zero on vertices v and equal there to $K(v) = 1 - S_0(v)/2 + S_1(v)/3$, where $S_0(v), S_1(v)$ are the number of vertices and edges in $S(v)$ for $v \in V$. In the case of a circular $S(v)$ we know $S_0(v) = S_1(v) = |S(v)|$ so that $K(v) = 1 - |S(v)|(1/3 - 1/2) = 1 - |S(v)|/6$. ∎

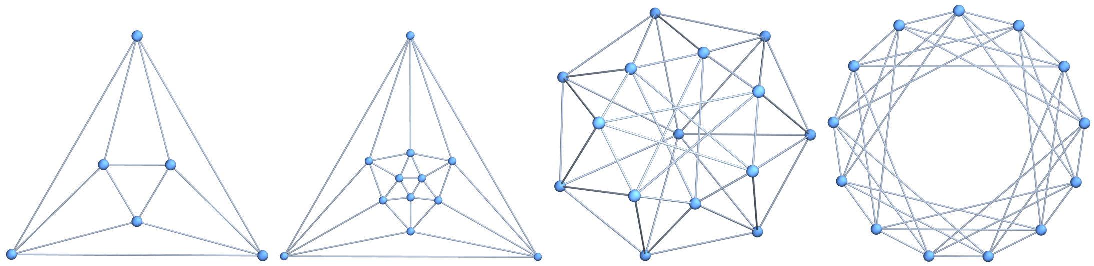

FIGURE 1. The **octahedron** has $K(v) = 1/3$ for all v. The **icosahedron** has $K(v) = 1/6$ for all v. A **projective plane** has curvatures in $\{0, 1/6, -1/6\}$. The flat torus = Clifford torus has constant 0 curvature.

11.5. An **edge collapse** $G \to G'$ takes $(a, b) \in E$ and identifies a with b. It removes 2 faces, 3 edges and 1 vertex so that $\chi(G) = \chi(G')$. A 2-manifold G is a **2-sphere** S^2, if $\chi(G) = 2$. The **connected sum** $G \# H$ of two 2-manifolds G, H is obtained by removing an edge in both manifolds and identifying the C_4 boundaries of the holes. If $v \in V(G)$ and $w \in V(H)$, with $|S(v)| = |S(w)|$, one can also remove v from G and w from H and glue boundaries to get a $G \# H$ with $\chi(G \# H) = \chi(G) + \chi(H) - 2$. A **2-ball** is a graph obtained from a S^2 by removing a vertex v. A **2-cylinder** or **handle** is a 2-sphere in which two vertices in distance > 2 removed. A **2-torus** is a 2-manifold obtained from a 2-cylinder by gluing the boundaries, matching orientation. A **Moebius strip** is a projective plane with one vertex removed. When glued into a hole of a sphere it is a **cap**. The **Klein bottle** is a S^2 with two caps. The projective plane is a sphere with a cap. The **boundary** of a G is $\{w \in V | S(v) \text{ is not a circle }\}$. The boundary of a ball or a Moebius strip is a circle. 2-manifolds have no boundary.

11.6. A **topological deformation** of a 2-manifold G takes a 2-ball in G and replaces it with an other 2-ball with the same boundary. In other words, a topological deformation is the process $G \to G' = G \# S^2$ implying $\chi(G) = \chi(G')$. Two 2-manifolds G, H are **topologically equivalent** if they can be deformed into each other by a finite set of topological deformations. An example of a topological deformation is to take out an edge and fill in the opposite diagonal edge. This **diagonal flip** is known as **Pachner transformation**. The following theorem is a milestone of 19'th century mathematics:

Theorem 2 (Classification of 2-manifolds)**.** *Every connected 2-manifold is equivalent to a 2-sphere S^2 or a connected g-sum of either $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ or $\mathbb{P}^2 \# \cdots \# \mathbb{P}^2$.*

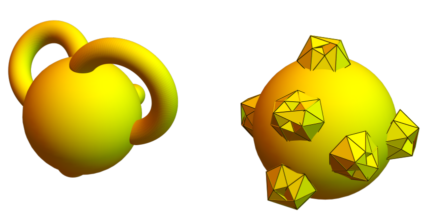

FIGURE 2. A 2-manifold is either S^2 with g handles (orientable) and $\chi = 2 - 2g$ or a S^2 with g cross caps and $\chi = 2 - g$ (non-orientable).