DIFFERENTIAL GEOMETRY

MATH 136

Lecture 10: Curvature

10.1. The **shape operator** S or **Weingarten map** encodes how the surface curves in $\mathbb{R}^3$. S maps the tangent space T_pM to the tangent space T_pS^2. It maps r_u to $-n_u$ and r_v to $-n_v$, meaning $Sdr^T = -dn^T$. By identifying T_pM with T_pS^2 it is a self-map of T_pM. In the basis $\{r_u, r_v\}$, it becomes a 2×2 matrix $A = S^T$ satisfying $\boxed{dn = -drA}$. Take this matrix equation for 3×2 matrices as the relation defining the shape operator. [1]

Theorem 1 (Shape operator)**.** *The shape operator matrix is* $\boxed{A = I^{-1}II}$.

Proof. Using the 3×2 matrices dr, dn we have defined A as $dn = -drA$. The second fundamental form is $II = -dr^T dn = dr^T drA = IA$. Since I is invertible, we can solve for A and get $A = I^{-1}II$. □

While A is not necessarily symmetric, it is symmetric with respect to the inner product $\langle v, w \rangle = v^T gw$. Proof $\langle Av, w \rangle = (Av)^T Iw = v^T A^T I = v^T II^T (I^{-1})^T Iw = v^T II^T w = v^T IIw = w^T IIv$ because II was symmetric. Having been able to switch v,w shows $\langle Aw, v \rangle = \langle w, Av \rangle$.

10.2. Define the **Gaussian curvature** as $\boxed{K = \det(A)}$. Written out, the curvature is

$$K = \det(A) = \frac{\det(II)}{\det(I)} = \frac{LN - M^2}{EG - F^2} = \lambda\mu \ .$$

From linear algebra, we know it is the product of the eigenvalues λ, μ of A. The **mean curvature** H is defined as the average of eigenvalues λ, μ of A. It is

$$H = \frac{\operatorname{tr}(A)}{2} = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{\lambda + \mu}{2} \ .$$

Theorem 2.

$$K = \frac{\det(II)}{\det(I)}$$

is independent of the basis.

Proof. We immediately have from the product determinant formula and the fact that I is invertible that $\det(K) = \det(A) = \det(II)/\det(I)$. Since determinants are independent of the basis, also the curvature is. □

[1] Einstein: $I = g_{ij}$ and $I^{-1} = g^{ij}$ and $II = h_{ij}$ and $A_i^k = g^{kj} h_{ji}$. The shape operator is a "linear transformation" $A_k^i v^k = w^i$ on vectors. I, II are quadratic forms **"(0,2) tensor fields"** while A is a transformation at every point, a **"(1,1) tensor field"**. $dn = -drA$ are called Weingarten equations.

10.3. We write $\iint_M f dV$ for the integral $\iint_R f(u,v)|r_u \times r_v| dudv$. For $f = 1$, this is the **surface area** $|M| = \iint_R |r_u \times r_v|\ dudv$. Since $n(u,v)$ parametrizes the unit sphere, we have $\iint_R |n_u \times n_v|\ dudv = 4\pi$. For convex surfaces, we can use the same parameter domain $R = [0, 2\pi) \times [0, \pi)$ and see that the total curvature is the same than the total curvature of a sphere. This requires that K is positive. The area of the image of S is called the **total curvature**. We have now already a cool version of Gauss-Bonnet: The general version will work for any surface, not only for convex (and so positive curvature) surfaces.

Theorem 3 (Gauss-Bonnet for convex closed surfaces)**.** *$\iint_M K dV = 4\pi$.*

Lemma 1. $\boxed{III = II^T A}$ *and so* $\det(III) = \det(A)^2 \det(I)$.

Proof. Start with the definition $dn = -drA$. Multiply with dn^T from the left to get $III = dn^T dn = -dn^T drA = -(dr^T dn)^T A = II^T A$. Taking determinants gives $\det(III) = \det(II)\det(A) = \det(I)\det(A)\det(A)$. □

10.4. The two identities $II = IA$ and $III = IIA$ can be used for a proof of the identity $\boxed{III - 2HII + KI = 0}$ without using the inner product defined by I. [2] Now to the proof of the Gauss-Bonnet result:

Proof. Take square roots of the lemma gives $\sqrt{\det(III)} = K\sqrt{\det(I)}$. This step has required K to be non-negative. Therefore,

$$\begin{aligned} 4\pi &= \iint_R |n_u \times n_v| dudv = \iint_R \sqrt{\det(III)}\ dudv \\ &= \iint_R K\sqrt{\det(I)}\ dudv = \iint_R K|r_u \times r_v| dudv = \iint K\ dV\ . \end{aligned}$$

□

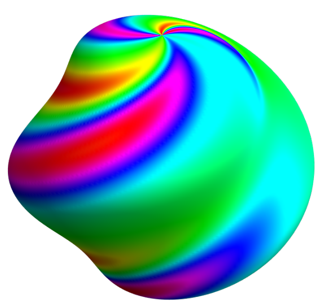

FIGURE 1. We see a convex surface colored with the curvature function K. Gauss-Bonnet establishes that the total curvature is 4π.

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[2]Thanks to some students of the course to point this out.