DIFFERENTIAL GEOMETRY

MATH 136

Lecture 9: Fundamental Forms

9.1. A surface M in $\mathbb{R}^3$ is defined by a C^2 map $r : R \to r(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ from a planar domain R to space $\mathbb{R}^3$. The partial derivatives r_u, r_v are tangent to the **grid curves** $u \to r(u,v)$ and $v \to r(u,v)$ and so tangent to M. If r is regular, the **unit normal vector** $n = r_u \times r_v / |r_u \times r_v|$ is defined and perpendicular to the surface.

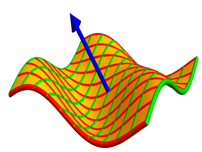

FIGURE 1. A parametrized surface $r(u,v)$ with unit normal vector $n(u,v)$. When seen as a map from M to S^2 it is known as the **Gauss map**.

9.2. We have already seen that the **first fundamental form** $I = g = dr^T dr$ satisfies $\det(I) = |r_u \times r_v|^2$.

Theorem: First fundamental form:

$$I = g = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$$

is a symmetric positive definite bilinear form.

Proof. The matrix dr is a 3×2 matrix and dr^T is a 2×3 matrix: $dr^T = \begin{bmatrix} - & r_u & - \\ - & r_v & - \end{bmatrix}$, $dr = \begin{bmatrix} | & | \\ r_u & r_v \\ | & | \end{bmatrix}$. The product is a 2×2 matrix. Now $g = dr^T dr$. We have seen already that $\det(g) = |r_u \times r_v|^2$. The trace of g is $\mathrm{tr}(g) = |r_u|^2 + |r_v|^2$. Having positive trace and positive determinant assures that we have a positive definite matrix. We call g a **bilinear form** because it maps two vectors X, Y to a number $\langle X, Y \rangle = X^T g Y$. It defines us a scalar product on the surface. $\square$

9.3. Examples:

1) In the case of a graph of a function $r(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$ we have $g = \begin{bmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{bmatrix}$.

2) In the sphere case $r(u,v) = \begin{bmatrix} \sin(v)\cos(u) \\ \sin(v)\sin(u) \\ \cos(v) \end{bmatrix}$ we have $g = \begin{bmatrix} \sin^2(v) & 0 \\ 0 & 1 \end{bmatrix}$. Note that at $v=0$ and $v=\pi$ this is not regular.

9.4. If $r : R \to \mathbb{R}^3$ is a regular C^2 parametrization of a surface M, define

$$n(u,v) = \frac{r_u \times r_v}{|r_u \times r_v|} .$$

It is continuously differentiable because r was assumed to be C^2. The Jacobean derivative dn is the 3×2 matrix $dn = \begin{bmatrix} | & | \\ n_u & n_v \\ | & | \end{bmatrix}$. We can combine it with dr^T and define the **second fundamental form** $h = -dr^T dn$. It agrees with $(d^2 r)^T n$.

Theorem: Second fundamental form

$$II = h = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} -r_u \cdot n_u & -r_u \cdot n_v \\ -r_v \cdot n_u & -r_v \cdot n_v \end{bmatrix} = \begin{bmatrix} n \cdot r_{uu} & n \cdot r_{uv} \\ n \cdot r_{vu} & n \cdot r_{vv} \end{bmatrix}$$

is a symmetric bilinear form.

Proof. From $r_u \cdot n = 0$ we get $r_{uu} \cdot n = -r_u \cdot n_u$ and similarly get $r_{uv} \cdot n = -r_u \cdot n_v$. Now, $II = -dr^T dn$ is symmetric because Clairaut applies. Clear is $r_u \cdot n_v = n_v \cdot r_u$. □

9.5. The **third fundamental form** is $III = e = dn^T dn$ is the first fundamental form of the sphere map n.

Theorem: Third fundamental form:

$$III = e = \begin{bmatrix} n_u \cdot n_u & n_u \cdot n_v \\ n_v \cdot n_u & n_v \cdot n_v \end{bmatrix} .$$

is a symmetric bilinear form and $|n_u \times n_v|^2 = \det(III)$.

Proof. $III = dn^T dn$ is symmetric as the dot product is commutative. The proof of $|n_u \times n_v|^2 = \det(III)$ is word by word identical what we have done in the second class for $r_u \times r_v$. □

9.6. The third fundamental form is not independent from the other two fundamental forms. In homework: with $H = \mathrm{tr}(A)/2$ and $K = \det(A)$ are trace and determinant of $A = I^{-1} II$:

Theorem: Compatibility: $III - 2H\,II + K\,I = 0$