To Lecture 8: Frenet inverse problem

8.1. Frenet's theorem describes regular curves $r(t)$ in $\mathbb{R}^n$ in terms of the rotation matrix $Q(t)$ that orients the Frenet frame as a function of t. Every Frenet curve is described as such. The reverse is to start with a path of rotation matrices $Q(t)$ and to ask about the properties of the curve. A curve of curvature matrices $K(t) \in so(n)$ leads via the differential equation $Q' = K(t)Q$ to a path in $Q(t) \in SO(n)$. The first row of $\int_0^t Q(s)\, ds$ is the curve $r(t)$. We have now $r'' = K$. One question can be. For which periodic paths $K(t)$ does the curve stay bounded? For which paths $K(t) + tK_0$ does the curve stay bounded?

8.2. Stability of driven systems is a classical problem, for example in celestial mechanics. Here is an other situation. If the curvature grows $K(t) + tK_0$ we get a continuous time version of the **Curlicue problem**. That problem is obtained by taking a sequence $K(t) + tK_0$ in $so(n)$ then look at $Q(t+1) = e^{iK(t)}Q(t)$ with $Q(0) = 1$. This produces **curlicue paths** $R(n) = \sum_{k=0}^{n-1} Q_k$ in the algebra of matrices. Paths can be seen by applying $R(n)$ to a vector. Both the Frenet or Curlicue problem naturally go over to manifolds. In the curlicue problem drive in each step along a geodesic path and in the continuum we get a space dependent Frenet problem $Q' = K(r(t), t)Q$, where now the curvature K is point dependent.

8.3. Let us now focus on a special case in the plane $n = 2$: you drive in a desert and $\kappa(t)$ tells you how to turn your steering wheel. Do you remain in a bounded region? For $n = 2$, we can use complex analysis. The function $\kappa(t)$ gives an explicit curve $r(t) = \int_0^t e^{i\phi(s)}\, ds$, where $\phi(s) = \int_0^s \kappa(u)\, du$ is the phase. We can investigate the case $\kappa(t) = k + a\sin(mt)$ for example We can think of the Frenet equations $\psi' = i\kappa(t)\psi$ as a Schrödinger equation. We prove here that stability happens if and only if k is not a multiple of the oscillator frequency.

8.4. In two dimensions, where $so(2) = \mathbb{R}$ we can ask for which periodic functions $\kappa(t)$ the curve $r(t) = \int_0^t e^{i\phi(s)}\, ds \in \mathbb{C}$ remains bounded, where $\phi(s) = \int_0^s \kappa(u)\, du$ is the phase. An example of a stable curve $\kappa(t) = 1 + \sin(2t)$. An example of an unstable one is $2 + \sin(2t)$. Also for $\kappa(t) = 1 + \sin(2t) + \sin(3t)$ we run away. The case $3 + \cos(2t) + \sin(4t)$ is stable but $2 + \cos(2t) + \sin(4t)$ is unstable. $\kappa[t_] := \sqrt{2} + \sin(2t)$ gives a bounded, non-closed curve.

Theorem 1. *A planar curve with curvature $\kappa(t) = k + a\sin(mt)$ is bounded if and only if k is not a multiple of m.*

8.5.

Proof. This integral is of the form $F(t) = \int_0^t e^{i\phi(s)} \, ds$ where ϕ is the phase function in the theory of stationary phase. For $\kappa(t) = k + am\sin(mt)$ we have is $\phi(s) = ks - a\cos(ms)$ requiring us to integrate $f(s) = \exp(iks)\exp(-ia\cos(ms))$ over $[0, 2\pi]$. Fourier expansion gives

$$f(s) = \sum_{n=-\infty}^{\infty} J_n(a)e^{i(k-mn)s}$$

so that

$$F(t) = \sum_{n=-\infty}^{\infty} J_n(a)(e^{i(k-mn)t} - 1)/(i(k - mn)) \ .$$

If $k - mn$ is never zero, then this is a bounded function. If $k + mn = 0$, Hopital shows that that $(e^{i(k-mn)t} - 1)/(i(k - mn)) = t$. The curve therefore grows like $2\pi(-i)^{k/m}J_{k/m}(a)$. $\qquad\square$

8.6. With more frequencies like $k + \sin(\alpha t) + \sin(\beta t)$ we have instability if and only if k is in the module generated by α, β. This means that for any 2π periodic trigonometric polynomial $p(t)$ we have stability for $\kappa(t) = k + p(t)$ for almost all k. But there is a dense set of k's for which we have instability.

8.7. There are many related questions. Here are more: A) What is the closure of the path $Q(t) \in SO(3)$. What are the α and ω limit sets?
B) When is the curve C defined by κ and τ bounded?
C) Is it possible that for almost periodic functions κ, τ the curve C in $\mathbb{R}^3$ has an ω limit set that is a fractal?
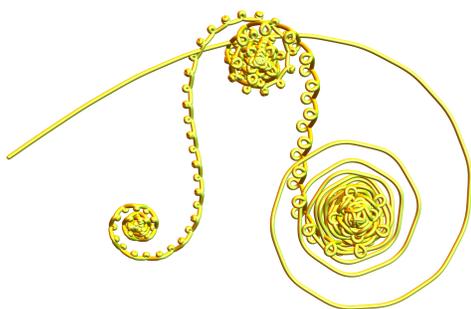

FIGURE 1. An example with $\kappa(t) = t\sin^2(t)/100, \tau(t) = 2\cos(3t)$.

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