

DIFFERENTIAL GEOMETRY

MATH 136

Lecture 8: Four vertex theorem

8.1. A **vertex** of a C^2 planar curve is a point $r(t)$ where the curvature function $\kappa(t)$ has a maximum or minimum. In the case of a non-circular ellipse, there are two maxima and two minima, and so 4 vertices in total. What happens in general? A curve is called **convex** if it bounds a convex region R . A region $R \subset \mathbb{R}^2$ is called **convex** if the line segment between any two points $A, B \in R$ is part of the region.

Theorem 1. *A simple closed regular convex C^3 plane curve has at least 4 vertices.*

Proof. We can assume κ is not constant. As a continuous function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has by the extremal value theorem at least one maximum a and one minimum b . We first show that we have more than just 2 extrema. So, assume, $r(a)$ and $r(b)$ were the only critical points. Choose the coordinate system so that these two points $r(a), r(b)$ are on the x -axis so that $r(t) = [x(t), y(t)]$ and $y(a) = y(b) = 0$. Convexity assures that there is no other root $y(t) = 0$. So, k' changes sign at $s = a$ and $s = b$ and nowhere else and $k'(s)y(s)$ does not change sign at all. The Frenet equations tell $e_1 = T = [x', y']$, $e_2 = N = [-y', x']$, $[x'', y''] = e_1' = \kappa e_2 = \kappa[-y', x']$ and imply that $x'' = -\kappa y'$. Now, we integrate over the entire curve $\int_0^L \kappa'(s)y(s) ds = \kappa y|_0^L - \int_0^L \kappa(s)y'(s) ds = \int_0^L x''(s) ds = x'(L) - x'(0) = 0$ (because C is closed). But note that $\kappa'(s)y(s)$ does not change sign since both κ' and y only change sign at two points. There is therefore an other maximum or minimum. The number of local max and local min are the same for a periodic function (they must alternate as two successive maxima have a minimum between) so that there must be 4 vertices. \square

8.2. Examples.

- 1) The ellipse $r(t) = [2 \cos(t), \sin(t)]$ has the curvature $\kappa(t) = 2(\cos^2(t) + 4 \sin^2(t))^{-3/2}$ which has maxima at $t = 0, \pi$ and minima at $t = \pm\pi/2$.
- 2) The curve $r(t) = 5[\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature $r'(t) \times r''(t) / |r'(t)|^3$ that has minima at $0, \pi$ and maxima at $\pm \arccos(-2/5)$.
- 3) The **limaçon** $r(t) = [\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature with only 2 vertices! Why is this not a counter example?

8.3. Remarks.

- 1) Convexity is not really needed. Osserman showed that for any simple closed C^2 curve C there are $2n$ vertices if the smallest circle enclosing C intersects it in at least n connected components. Also, a general simple closed C^2 curve has at least 4 vertices.
- 2) V. Arnold conjectured that the result holds for any curve that can be obtained from

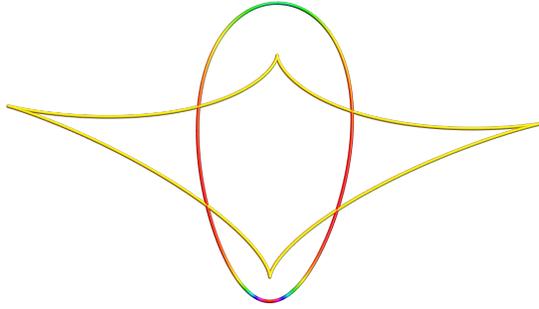


FIGURE 1. A simple closed curve for which curvature is coded as color. The 4 vertices are visualized by plotting the **evolute** $e(t) = r(t) + n(t)/\kappa(t)$, where $n(t)$ is the normal vector pointing inside. The vertices of the curve correspond to cusps of the evolute.

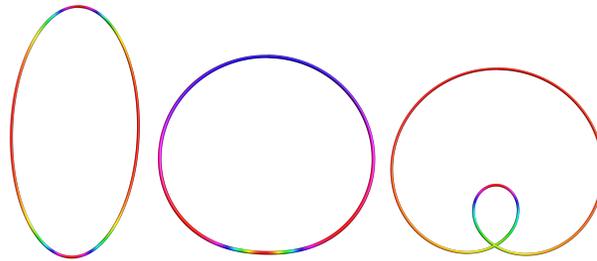


FIGURE 2. The ellipse, a deformation of an ellipse and a limaçon.

a circle by suitable deformations.

3) Converse result: If a continuous real valued, periodic function has at least two local maxima and two local minima, it is the curvature function of a simple closed curve.

8.4. History.

1) The theorem was proven in 1909 by Syamadas Mukhopadhyaya for convex curves. 2) The general case was published in 1912 by Adolph Kneser. 3) The proof given here in the convex case is due to G. Herglotz in 1930. 4) Robert Osserman in 1985 generalized the result to the "four or more vertex theorem". 5) The converse result started with H. Gluck in 1971 and was proven in 1997 by Bjoern Dahlberg.

8.5. Related.

1) The **evolute** of a plane curve is defined as $e(t) = r(t) + n(t)/\kappa(t)$. It is the caustic of the normal map. At points, where $\kappa'(t)$ is zero, the evolute has **cusps**. A caustic of a simple closed curve therefore has at least 4 cusps. For the ellipse the evolute is called the Lamé curve.

2) The **tennis ball theorem** states that a C^2 curve on the sphere that divides the sphere into regions of equal area have at least 4 inflection points.

3) The open **last geometrical problem of Jacobi** asks whether a caustic on an ellipse has at least 4 cusps.