DIFFERENTIAL GEOMETRY

MATH 136

Lecture 7: Hopf Umlaufsatz

7.1. The theorem of today deals with **signed curvature** $\kappa = \frac{r' \times r''}{|r'|^3}$ using the cross product in 2 dimensions. [1]. We do not assume today that the curve is Frenet. The curvature is allowed to become zero. We assume however that the curve is regular, meaning that $dr = r'$ is never zero, as well as closed. We have seen that there is then an arc length parametrization of the curve and $|\kappa| = |r''|$ because $r' \cdot r' = 1$ implies r'' is perpendicular to r'. We deal with the last curvature and so torsion which always can have signs.

7.2. If the curve is parametrized on $[a, b]$, the **rotation index** is defined as $\frac{1}{2\pi} \int_a^b \kappa(t)\, dt$. In general, if the closed curve is not arc length parametrized, the rotation index is defined as $\boxed{\int_a^b \kappa(t)|r'(t)|\, dt}$.

Theorem 1. *The rotation index of a closed C^2 curve is in $\mathbb{Z}$.*

Proof. Using arc length parametrization, write

$$r'(t) = [\cos(\alpha(t)), \sin(\alpha(t)]$$

then $\kappa = \alpha'$ (we did that more detailed in class). Since the curve is closed, we have $\alpha(b) - \alpha(a) = 2\pi n$, where n is an integer. ☐

7.3. The case $r(t) = [\cos(nt), \sin(nt)]$ with $t \in [0, 2\pi]$ shows that the rotation index can take any integer value n. It is intuitively clear that if a curve has no self intersections, then the index must be either 1 or -1. This is not so obvious however. We do not want for example to refer to the Jordan curve theorem, telling that a continuous simple closed curve in the plane divides the plane into an inside and outside. Heinz Hopf found a nice argument which proves this "Umlaufsatz" (rotation angle theorem) in an elegant way using a deformation picture:

Theorem 2 (Hopf Umlaufsatz)**.** *A simple closed regular C^2 curve has rotation index 1 or -1.*

Proof. We assume that $r(t)$ is parametrized on $[0, 1]$ and parametrized by arc-length. Define on the square $Q = [0, 1] \times [0, 1]$ the function $f : Q \to \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\}$ given by $f(t, s) = \arg((r(t) - r(s))/|r(t) - r(s)|)$ for $t \neq s$ and $\alpha(t) = f(t, t) = \arg(r'(t)/|r'(t)|)$ for $t = s$. Because $r \in C^1$, the function f is continuous on $Q = [0, 1] \times [0, 1]$. Now comes

[1]The cross product in n dimensions has $\binom{n}{2} = n(n-1)/2$ components. For $n = 2$ it is a scalar

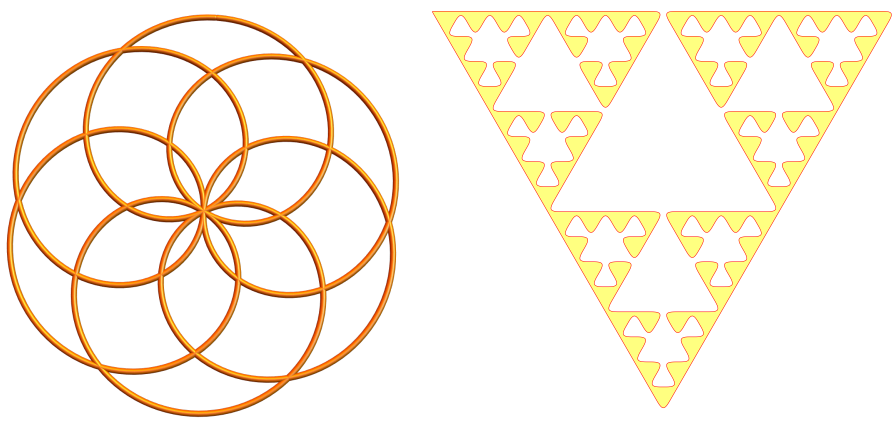

FIGURE 1. To the left, we see $r(t) = [\cos(t)+\cos(7t), \sin(t)+\sin(7t)]$, $t \in [0, 2\pi]$ which looks like the flower of life. Its rotation number is 7. We can compute $\kappa(t)|r'(t)| = 4 + 72/(25 + 8\cos(6t))$ which integrates on $[0, 2\pi]$ up to 14π. To the right, a simple closed smooth curve in the plane. What is its rotation number?

a **homotopy argument**. The index is $[f(1,1) - f(0,0)]/(2\pi)$ and so an integer. If we move along the diagonal and look at $\alpha(t) = f(t,t)$ we see a continuous curve which moves on the circle $\mathbb{T}$. If we deform the curve, the total change remains the same. We can continuously deform the parameter curve so that we first deform from $(0,0)$ straight to $(0,1)$ and then straight from $(0,1)$ to $(1,1)$. Choose a coordinate system so that is in $y \geq 0$ just touching the x-axes. If $r'(0) = [a, 0]$ with positive a then $f(t,s) \in [0,\pi]$ with $f(0,0) = 0$ and $f(0,1) = \pi$ and then $f(1,1) = 2\pi$. If $a < 0$, then $f(t,s) \in [-\pi, 0]$ with $f(0,0) = \pi$ and $f(0,1) = 0$ and then $f(1,1) = -\pi$. In the former case, $i = 1$ in the later $i = -1$. $\square$

7.4. Remarks:
1) This is a Gauss-Bonnet type result for a flat 2-manifold with boundary.
2) The proof shows that this even works for C^1 curves as $f(t,t) - f(s,s)$ is just the angle change of the tangent. This works even if the curvature is not defined. In the homework you will push it to polygons. Most texts assume C^2.
3) Closed C^0 curves are trickier: Jordan curve theorem, open peg problem of Toeplitz.

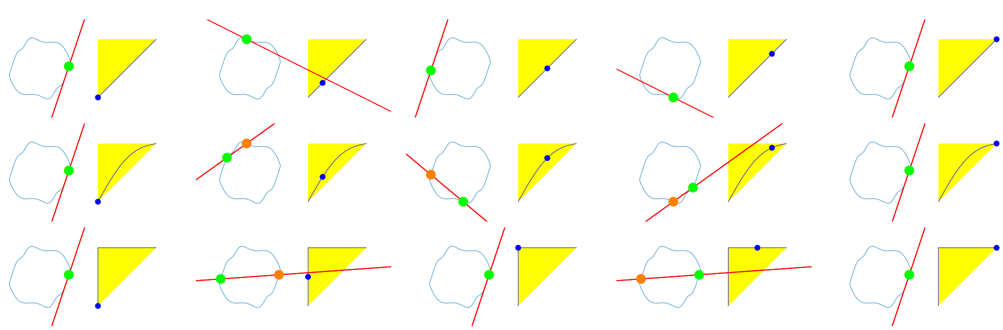

FIGURE 2. The deformation argument.