DIFFERENTIAL GEOMETRY

MATH 136

Lecture 6: Fundamental theorem of curves

6.1. A **Frenet curve** is given by a C^n map $r : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ for which $r', r'', \dots, r^{(n)}$ are linearly independent at every point. In the case $n = 3$ we had seen this as $r' \times r'' \neq 0$. Let $e_1, e_2, \dots, e_n$ denote the orthonormal frame obtained by Gram-Schmidt. One can get this as follows: build the matrix R with $r', r'' \dots, r^{(n)}$ as rows and perform the QR decomposition to get an orthonormal matrix Q in which the vectors $e_1, e_2, \dots, e_n$ are the rows. [1] Define the curvatures $\boxed{\kappa_j = e_j' \cdot e_{j+1}}$. It is positive for $1 \leq j \leq n - 2$. The largest κ_{n-1} is also called the **torsion** and is not necessarily positive. A natural generalization of the Frenet formulas to arbitrary dimensions is

Theorem 1 (Frenet-Serret formulas)**.**

$$\begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ \cdots \\ e_{n-1} \\ e_n \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \cdots & \cdots \\ 0 & -\kappa_2 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & \kappa_{n-1} \\ 0 & \cdots & \cdots & 0 & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ \cdots \\ e_{n-1} \\ e_n \end{bmatrix} .$$

Proof. To get the entries of K expand e_j' in terms of the $e_1, \dots, e_n$.

$$e_j' = \sum_{i=1}^{n} (e_j' \cdot e_i) e_i \ .$$

This means $Q' = KQ$ with skew symmetric K. Especially, the diagonal entries of K are zero. The skew symmetry can be seen from $e_j \cdot e_k = 0$ for all $j \neq k$ implying $e_j' \cdot e_k = -e_j \cdot e_k'$. For every $j \leq n-1$, the e_j by definition are in the subspace generated by $r', r'', \dots, r^{(j)}$ which is the subspace generated by $e_1, \dots, e_j$ and e_j' therefore generated by $e_1, \dots, e_j$. This implies $e_j' \cdot e_{j+2} = e_j' \cdot e_{j+3} = \cdots = e_j' \cdot e_n = 0$. The only entry in the upper triangular part is $(e_j' \cdot e_{j+1}) = \kappa_j$. $\square$

6.2. One can check the skew symmetry of K abstractly starting with $Q^T Q = 1$. A fancy way to restate is that in the Lie group $SO(n)$, the tangent space is the Lie algebra $so(n)$.

[1] Frenet and Serret have discovered the $n = 3$ dimensional case independently. The higher dimensional case has appeared only in the 20th century.

Lemma 1. *If $Q(t)$ is a curve of orthogonal matrices, then $Q'(t) = K(t)Q(t)$ with skew symmetric K.*

6.3. Given curvatures $\kappa_1(t) > 0, \dots, \kappa_{n-2}(t) > 0, \kappa_{n-1}(t)$ which are all continuous, we get a continuous path $K(t)$ of skew symmetric matrices.

Theorem 2 (Fundamental theorem of curves)**.** *Given curvatures κ_j, there is up to translation and rotation a unique Frenet curve which has these curvatures.*

Proof. The curvatures define a curve $K(t)$ of skew symmetric matrices. The differential equation $Q' = K(t)Q = F(t, Q)$ is linear in Q and so smooth. Since the solution of this differential equation gives orthogonal matrices $Q(t)$ (check it!) the solution exists for all times. Proceed as in the 3 dimensional case by writing $r(t) = r(0) + \int_0^t r'(s)\, ds$ where $r'(s) = Q(s)_1$ is the first row of $Q(s)$. □

6.4. Examples.
1) If K is constant, then e^{Kt} solves $Q' = KQ$.
2) If K is constant and $n = 3$, then the curve is a spiral if $\tau \neq 0$ and a circle if $\tau = 0$.
3) In $\mathbb{R}^3$, the torsion is constant zero if and only if the curve is contained in a plane.
4) In $\mathbb{R}^n$ the torsion is constant zero if and only if the curve is contained in a $(n-1)$ dimensional hyperplane.
5) A line is not a Frenet curve and the above does not apply.
6) For non-Frenet curves, lots of things can go wrong. Assume for example, you have a curve which contains some part which is a line. While traveling along that line, we can turn around and lose track of the Frenet frame.

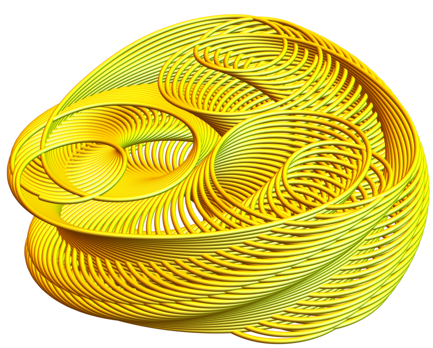

FIGURE 1. It can be fun to watch curve with simple $\kappa(t) = a + b\cos(ct), \tau(t) = d\sin(et)$

6.5. A famous example is the Euler curve. It is a plane curve for which $\kappa(t) = t$ is fixed. Here is a possible open research problem. Given periodic functions $\kappa(t) > 0, \tau(t)$ not necessarily with the same period. Under which conditions is the resulting curve r bounded when looking at $t \in \mathbb{R}$?

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2025