DIFFERENTIAL GEOMETRY

MATH 136

Lecture 5: Frenet Equations

5.1. Today, we work at smooth curves $r : [a, b] \to \mathbb{R}^3$. [1] Define the **unit tangent vector** $T(t) = r'(t)/|r'(t)|$, the **normal vector** $B(t) = T'(t)/|T'(t)|$ and the **binormal vector** $B(t) = T(t) \times N(t)$. The three vectors are defined, as long as as r' and T' are both non-zero. One calls it a **Frenet frame** (T, N, B). A smooth curve is called a **Frenet curve** if r', r'' are linearly independent at every t. This is equivalent to the statement $\boxed{r' \times r'' \neq 0}$ for every t.

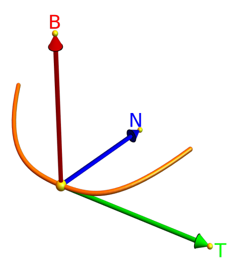

FIGURE 1. Direction T, normal N and binormal direction B.

Theorem 1. *For a Frenet curve, the Frenet frame TNB is orthonormal at every point.*

Proof. By assumption, r' and r'' are both not zero. If the parametrization is arc length, then $T = r'$ and $N = T'/|T'| = r''/|r''|$. Since $T \cdot T = 1$, we have by Leibniz product rule, $2T' \cdot T = 0$ so that N is perpendicular to T. The cross product $T \times B$ now also has length 1 and is perpendicular to both T and N. □

5.2. Every Frenet curve $r(t)$ can be parametrized by arc length as you work out in the homework. The **curvature** κ is then defined as $\boxed{\kappa = T' \cdot N}$ which is $|T'|$. The curvature measures the deviation of the curve from being linear. The **torsion** τ is defined as $\boxed{\tau = N' \cdot B}$. It measures the deviation from the curve of being planar. We can encode the three vectors T, N, B by turning them into row vectors of an **orthogonal** 3×3 **matrix** $Q(t) = \begin{bmatrix} T & N & B \end{bmatrix}^T$. We get now $Q'(t) = K(t)Q(t)$, where $K(t)$ is skew-symmetric:

[1] In $\mathbb{R}^3$, one requires the map $r : [a, b] \to \mathbb{R}^3$ to be at least C^3.

5.3.

Theorem 2 (Frenet equations)**.** $\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$

Proof. Because $N \cdot T = 0$, we have $N' \cdot T = -T' \cdot N = -\kappa$. The relation $N \cdot N = 1$ implies $N' \cdot N = 0$. The relation $B \cdot B$ implies $B' \cdot B = 0$. Now expand with respect to the basis $\{T, N, B\}$ and use $\kappa = |T'|$ and $\tau = N' \cdot B$:

$$\begin{aligned} T' &= (T' \cdot T)T + (T' \cdot N)N + (T' \cdot B)B = 0 + 0 + \kappa N \\ N' &= (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B = -\kappa T + 0 + \tau B \\ B' &= (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B = 0 - \tau N + 0 \,. \end{aligned}$$

□

5.4. In the two dimensional case, we only have to consider T and N. We can reduce to the planar case if τ is constant 0. The Frenet equations can then be written as $\begin{bmatrix} T \\ N \end{bmatrix}' = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}$.

5.5. The **fundamental theory of curves** in $\mathbb{R}^3$ tells that curvature and torsion determines a curve up to Euclidean congruences given by rotations or translations.

"The shape of a Frenet curve is determined by curvature and torsion".

Lemma 1. *For any differentiable curvature and torsion functions $\kappa(t) \geq 0$ and $\tau(t)$, there exists up to translation and rotation a* **unique** *curve $r(t)$ parametrized by arc length that has the given curvature and torsion.*

Proof. The initial $r(0)$ and $(T(0), N(0), B(0))$ fixes the initial point and location. Now "build the curve": the functions $\kappa(t), \tau(t)$ define a **skew symmetric** matrix $K(t) = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$. We define orthogonal matrix $Q(t)$ satisfying the differential equation $Q'(t) = K(t)Q(t)$. In the homework you will verify that a differential equation $x' = F(t, x)$ with a C^1 function F has locally a unique solution and that if $x(t)$ stays bounded, the solutions exist for all times. You also will check that if $Q(t)$ is a curve of orthogonal matrices, then $Q' = KQ$ with skew symmetric K. This can be reversed: if $Q(0)$ is orthogonal and $K(t)$ is skew symmetric, then the solution $Q(t)$ of the differential equation is orthogonal. Having $Q(t)$, it produces $r'(s) = Q(s)r'(0)$ and so $r(t) = r(0) + \int_0^t r'(s)\, ds$, where $r'(t) = r'(0) + \int_0^t Q^T(s) r'(0)\, ds$. □

5.6. We have mentioned in the warm-up class expressions for curvature and torsion for a curve $r(t)$. These formulas worked if the curve was not necessarily arc-length parametrized. In the Frenet case, meaning that $r' \times r'' \neq 0$, we will prove them in class:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}, \quad \tau(t) = \frac{\det[r'(t), r''(t), r'''(t)]}{|r' \times r''|^2} \,.$$

In class we will look at examples where we can compute the curvatures and torsion. A case where κ is arc length and a case where $\kappa = \tau$.