DIFFERENTIAL GEOMETRY

MATH 136

Lecture 3: Surfaces

3.1. Geometric objects can be given as **level sets**, kernels $\{f = 0\}$ of smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$ with $n < m$ or **parametrizations**, images of smooth maps f from a subset R of $\mathbb{R}^m \to \mathbb{R}^n$ with $m < n$. If in the level set case, df has maximal rank n everywhere, we get a **manifold**. [1] The same happens in the parametrization case, if f is injective and df has maximal rank m everywhere.

3.2. An example of **level surface** $\{f = 0\}$ of a function $f : \mathbb{R}^3 \to \mathbb{R}$ with $df \neq 0$ everywhere is the sphere $x^2 + y^2 + z^2 - 1 = 0$. An other example is a curve, the image of an interval $[a, b]$ to $\mathbb{R}^n$. The duality between kernel and image manifests already in linear algebra. The **kernel** $\ker(A)$ of a matrix A is the linear space $\{Ax = 0\}$ The **image** $\mathrm{im}(A)$ is the linear space $\{Ax\}$. The **fundamental theorem of linear algebra** is the wonderful duality $\boxed{\mathrm{im}(A^T) = \ker(A)^\perp}$.

Theorem: The image of A^T is perpendicular to the kernel of A.

Contour surfaces

3.3. If $f : \mathbb{R}^n \to \mathbb{R}$ is given, then the solutions of $f(x_1, \cdots, x_n) = d$ is called **hyper surface** or simply **surface** if $n = 3$. If the Jacobian matrix df (or equivalently the gradient $\nabla f = df^T$ is non-zero (meaning has maximal rank at every point), then $f = d$ is an example of a manifold. We will give more definitions later.

3.4. The case $f(x) = Ax$ is a hyperplane. **Quadratic manifolds** are $f(x) = x \cdot Bx + Ax = d$, where B is a symmetric matrix, A is a row vector and $d \in \mathbb{R}$ and df has maximal rank. Write $\mathrm{Diag}(a_1, \dots, a_n)$ for diagonal and 1 for the identity matrix.

3.5. Examples: For $B = 1$ and $A = 0$ and $d = 1$ we get the **sphere** $|x|^2 = 1$. For $B = \mathrm{Diag}(1/a^2, 1/b^2, 1/c^2)$ is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ we get ellipsoids. For $B = \mathrm{Diag}(1, 1, -1)$ and $d = 1$, we get a **one-sheeted hyperboloid** $x^2 + y^2 - z^2 = 1$. For $B = \mathrm{Diag}(1, 1, -1)$ and $d = -1$, we get a **two-sheeted hyperboloid** $x^2 + y^2 - z^2 = -1$. For $B = \mathrm{Diag}(1, 1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **paraboloid** $x^2 + y^2 = z$, for $B = \mathbf{Diag}(1, -1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **hyperbolic paraboloid** $x^2 - y^2 = z$. We can recognize paraboloids by intersecting with $x = 0$ or $y = 0$ to see parabola. If $B = \mathrm{Diag}(1, 1, -1)$ and $d = 0$, we get a **cone** $x^2 + y^2 - z^2 = 0$. For $B = \mathrm{Diag}(1, 1, 0)$ and $d = 1$ we get the **cylinder** $x^2 + y^2 = 1$.

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[1] A theorem of Nash assures that every m-manifold can be embedded in some $\mathbb{R}^n$.

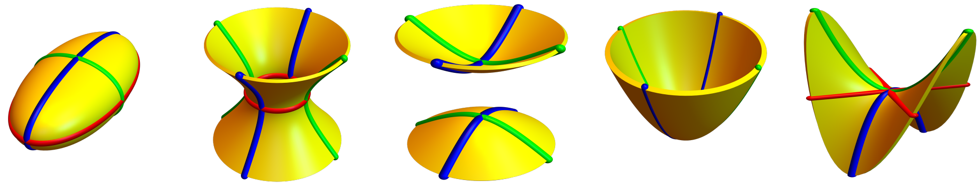

FIGURE 1. Ellipsoid, hyperboloids and paraboloids.

PARAMETRIZATIONS

3.6. A map $r : \mathbb{R}^m \to \mathbb{R}^n$ is called a **parametrization**. It is custom to use the letter r here, rather than f. We take the case $m < n$ and especially $m = 2, n = 3$. A map r from $\mathbb{R}$ to $\mathbb{R}^n$ is a **curve**. The image of a map $r : R \subset \mathbb{R}^m \to \mathbb{R}^n$ is then a **m-dimensional surface** in $\mathbb{R}^n$.

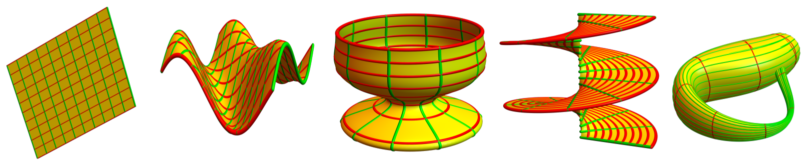

FIGURE 2. A plane, graph, surface of revolution, helicoid and Klein bottle

3.7. The parametrization $r(\phi, \theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ produces the **sphere** $x^2 + y^2 + z^2 = 1$. The full sphere uses $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$. By modifying the coordinates, we get an **ellipsoid** $r(\phi, \theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$ satisfying $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By allowing a, b, c to be functions of ϕ, θ we get "bumpy spheres" like $r(\phi, \theta) = (3 + \cos(3\phi)\sin(4\theta))[\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$.

3.8. If $r : \mathbb{R}^m \to \mathbb{R}^n$ with $m < n$ is a parametrization, then **Jacobian matrix** $dr(x)$ produces the $m \times m$ matrix with $\boxed{g = dr^T dr}$. It is the **first fundamental form**. For a parametrization $R : \mathbb{R}^2 \to \mathbb{R}^3$, the 3×2 matrix $dr(u, v)$ contains the vectors $\partial_u r = r_u, \partial_v r = r_v$ as columns and g is a 2×2 matrix.

3.9. The number $\boxed{|dr| = \sqrt{\det(dr^T dr)}}$ is called the **volume distortion factor**. The integral $\boxed{\int_R |dr(x)|\, dx}$ is the m-dimensional volume of the images $r(R) \subset \mathbb{R}^n$.

3.10. For a surface in $\mathbb{R}^3$, the surface area is $\boxed{\int\int_R |r_u \times r_v|\, dudv}$ because

Theorem: $\det(dr^T dr) = |r_u \times r_v|^2$ for $r : \mathbb{R}^2 \to \mathbb{R}^3$.

Proof. As $dr^T dr = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$, the identity is the **Cauchy-Binet identity** $|r_u \times r_v|^2 = |r_u|^2 |r_v|^2 - |r_u \cdot r_v|^2$ which boils down to $\sin^2(\theta) = 1 - \cos^2(\theta)$, where θ is the angle between the tangent vectors r_u and r_v. $\square$