DIFFERENTIAL GEOMETRY

MATH 136

Lecture 2: Preliminaries

2.1. A map

$$f : \mathbb{R}^m \to \mathbb{R}^n, \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} \mapsto \begin{bmatrix} f_1(x_1, \dots x_m) \\ \dots \\ f_n(x_1, \dots, x_m) \end{bmatrix}$$

is called **differentiable** or C^1, if all derivatives $\frac{\partial}{\partial x^j} f_i$ are continuous. For such a map, define the $n \times m$ **Jacobean matrix**

$$df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_1(x_1, \dots, x_m) \\ & \dots & \\ \frac{\partial}{\partial x_1} f_n(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_n(x_1, \dots, x_m) \end{bmatrix} .$$

2.2. a) If $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a curve, then $dr(t) = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ is the **velocity**.
b) If $f(x, y, z)$ is a function of 3 variables, then $df(x, y, z) = [f_x, f_y, f_z]$ is a 1×3 matrix. It is the transpose of the gradient.
c) For a vector field $f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$, we have $df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$. It is used also when analyzing systems of differential equations $x' = f(x)$.
d) If $f(x) = Ax$ is a linear map, then $df(x) = A$.

2.3. We say df has **maximal rank** if its rank is the minimum of m, n. If $m < n$, then the image of f is in general a m-dimensional set in $\mathbb{R}^n$. If $n < m$, then we can look at the roots $f(x) = 0$ which is a $n - m$ dimensional set. If $m \leq n$, we can say that f is **manifold like** near $f(x)$ if $df(x)$ has maximal rank. If $n \leq m$, then $M = \{f - c = 0\}$ is **manifold like** near x, if $df(x)$ has maximal rank. In a homework you review the fundamental theorem of linear algebra $\ker(A^T) = \mathrm{im}(A)^\perp$.

2.4. Example: $m = 2, n = 1$ and $f(x, y) = x^3y^2 - xy^3 = -4$. What happens near $(x_0, y_0) = (1, 2)$. To investigate this, we look at the Jacobean matrix $df(x, y) = [3x^2y^2 - y^3, 2x^3y - 3xy^2]$ which is at $(1, 2)$ equal to $[4, -8]$. The line $4x - 8y = -12$ is tangent to the curve. If we write the curve as $y = g(x)$ near $x = 1$, then $f(x, g(x)) = -4$ and differentiating gives $f_x + f_y g' = 0$ so that $g' = -f_x/f_y$. The next theorem assures that $g(x)$ exists.

Theorem 1 (Implicit function theorem)**.** *If $f(x, y) = f(x_0, y_0) = c$ and $f_y(x_0, y_0) \neq 0$, then $f(x, y) = c$ can near (x_0, y_0) be written as $y = g(x)$ for some C^1 function $g(x)$.*

Proof. Take a small neighborhood $U = I \times J$ of (x_0, y_0), where $|f_y(x,y)| \geq c$ and $|f_x(x,y)| \leq d$. Given $(x,y) \in U$ and $f(x_0,y_0) = 0$ and $y \to f_y(x,y)$ is bounded away from 0, we have $f(x, y_0 - t)f(x, y_0 + t) < 0$ and by the **intermediate value theorem**, there exists for x close to x_0 a $t = t(x)$ such that $f(x, y_0 + t(x)) = 0$. This gives us a function $g(x) = y_0 + t(x)$ and $f(x,y) = c$ agrees with $y = g(x)$ in U. By the chain rule $\frac{\partial}{\partial x} f(x, g(x)) = f_x \cdot 1 + f_y \cdot g' = 0$, we see that g is differentiable with $g' = -f_x/f_y$ and $|g'(x)| \leq d/c$ is bounded in U. $\square$

2.5. It follows that if a C^1 function f is manifold like near (x_0, y_0) then the level set $f(x,y) = c$ is near (x_0,y_0) the graph of a function: Proof: if $f_y \neq 0$ use the implicit function theorem giving $y = g(x)$. If $f_y = 0$, we have $f_x \neq 0$ and $f(x,y) = c$ is the graph of $x = g(y)$ for some C^1 function g.

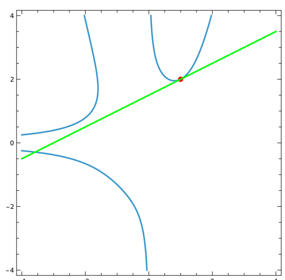

FIGURE 1. $f(x,y) = c$ can near (x_0, y_0) be written as a graph $y = g(x)$.

2.6. If $m < n$, then f has maximal rank if df has rank m. We look at this case more next week and look at $g = df^T df$. Lets look at a C^1 curve C defined by the parametrization $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Its velocity if $dr = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$. If the velocity is not zero at $t = t_0$ then the curve is manifold like. The curve is then close to the line $l(s) = r(t_0) + sr'(t_0)$. If $r'(t_0) \neq 0$, then C is a graph $(s, g(s))$ close to the line $(s, l(s)$.

2.7. In the case $m = n$, the map $f : \mathbb{R}^n \to \mathbb{R}^n$, the image of f is manifold like near x if $\det(df) \neq 0$. In that case f is invertible near x:

Theorem 2 (Inverse function theorem)**.** *If $h(y) \in C^1$ and $h'(y_0) \neq 0$, then h is invertible near y_0 and $y = g(x)$ near x_0. Also, g is C^1 with $dg(x_0) = dh(y_0)^{-1}$.*

Proof. Define $f(x,y) = x - h(y) = 0$. As $f_y = -h'(y)$ is non-zero, the theorem applies: there is $g(x)$ with $y = g(x)$ near x_0. It is the inverse: $x = h(y)$ and $y = g(x)$. $\square$

2.8. All proofs can be generalized to maps $f : \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto f(x,y) \in \mathbb{R}^n$. Now f_x is a $n \times k$ matrix and f_y is a $n \times n$ matrix. If f_y is invertible, then df has maximal rank and $f(x,y) = c$ can be written as $y = g(x)$ and $n \times k$ matrix $dg = -f_y^{-1} f_x$. This is the implicit function theorem. If $k = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, then $df(x)$ invertible implies f is invertible near x.

2.9. A point $x \in \mathbb{R}^m$ is a critical point of f if $df(x)$ does not have maximal rank. The value $f(x)$ is then called a **critical value**. In the homework you look up the Sard theorem assuring that for smooth f almost all values are critical values.